التراص من نوع α-C-

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الخلاصة

قمنا في هذا البحث بتعريف نوع جديد من التراص اسميناه "التراص من نوع α-c" كذلك قمنا بدراسة بعض خواصه والعلاقة بينه وبين التراص والتراص من نوع -α والتراص من نوع -c.

α - C-Compactness

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Abstract

In this paper, we introduce a new type of compactness which is called "α-c-compactness". Also, we study some properties of this type of compactness and the relationships among it and compactness, α-compactness and c-compactness.

1. Introduction and Preliminaries

A topological space (X,τ) is said to be c-compact space if for each closed set $A \subseteq X$, each open cover of A contains a finite subfamily W such that $\{cl\ v: v \in W\}$ covers A, [1].

In 1965, O.Njasted [2] introduced "α-open set" in topology [A subset A of a topological space X is said to be "α-open set if $A \subseteq int(cl(int(A)))$], and he proved that the family of all "α-open sets in a space (X,τ) is a topology on X, which is finer than τ and denoted by τ_α.

α-open sets are discussed in [3], [4], [5], some concepts were studied as follows:

i. The complement of an α-open set is called α-closed set and the intersection of all α-closed sets contains a set A which is called the α-closure of A and denoted by α-clA. So, α-clA is an α-closed set and proved (α-clA = A iff A is α-closed set).

ii. If A be a subset of a topological space X the α-derived of A is the set of all elements x satisfies the condition, that for every α-open set V contains x, implies $V\backslash\{x\}\cap A \neq \emptyset$.

In 1985, the term of "α-compactness" was used for the first time by S.N.Maheshwari and Thakur [6]. A space X is called α-compact space if every α-open cover for X has a finite subcover.

In this paper we shall introduce a new concept of compactness, which is called an "α-c-compactness" where [A topological space X is said to be α-c-compact space if for every α-closed set $A \subseteq X$, each family of α-open sets in X which covers A, there is a finite subfamily W such that $\{\alpha\text{-cl}\ U : U \in W\}$ covers A].

We discuss some properties of this kind of compactness and give some propositions, corollaries and examples After investigating the relationships among compact spaces,c-compact spaces, α-compact spaces and α-c-compact spaces are considered.

1.1 Definition [1]

A topological space (X,τ) is said to be c-compact if for each closed set $A \subseteq X$, each open cover of A contains a finite subfamily W such that $\{cl\ v: v \in W\}$ covers A.

1.2 Proposition [1]

Every compact space is c-compact.

1.3 Remark

The implication in proposition (1.2) is not reversible, for example:

A space (N,τ) where, $\tau = \{U_n = \{1,2,\ldots,n\} \mid n \in N\} \cup \{N,\emptyset\}$ is c-compact which is not compact.

1.4 Proposition [1]

A T_3-c-compact space is compact.

1.5 Definition [6]

A space X is said to be α-compact space if every α-open cover of X has a finite subcover.

1.6 Proposition [6]

Every α-compact space is compact.

1.7 Remark

The opposite direction of proposition (1.6) may be false, for example:

Let X = {0} ∪ N and τ = {∅,{0},X} be a topology on X. Evidently, X is a compact space. However, it is not α-compact space.

1.8 Proposition [6], [7]

If all nowhere dense subsets of a topological space X are finite, then the concepts of compactness and α-compactness are concident.

In propositions (1.9) and (1.11) we shall discuss the relationships between α-compatness and c-compactness.

1.9 Proposition

Every α-compact space is c-compact.

Proof:

Follows directly from propositions (1.6) and (1.2).

1.10 Remark

The opposite direction of proposition (1.9) may be false, see the example in remark (1.3), (N,τ) is c-compact space which is not α-compact, since {{1,n} | n ∈ N} is α-open cover for N which has no finite subcover.

1.11 Proposition

If all nowhere dense subsets of a T_3- space X are finite, then X is α-compact space, whenever it is c-compact..

Proof:

Follows from propositions (1.4) and (1.8).

2. α-c-compactness

2.0 Introduction

In this section we shall introduce a new type of compactness which is termed "α-c-compactness", we shall study further properties of this type of compactness. Examples were constructed to show the relationships among "compact, c-compact, α-compact and α-c-compact space". Several propositions of these spaces are given also

2.1 Definition

A topological space (X,τ) is said to be α-c-compact space if for each α-closed set A ⊆ X, each family of α-open subset of X which covers A has a finite subfamily whose α-closures in X covers A.

2.2 Proposition

An α-compact space is α-c-compact.

Proof:

Let A be an α-closed subset of an α-compact space X and $\{U_\lambda:\lambda \in \wedge\}$ be a family of α-open sets in X which covers A, implies, $\{U_\lambda:\lambda \in \wedge\}$ ∪ {X – A} is an α-open cover of X which is α-compact space, then there is a finite family $\{U_{\lambda_i}:i = 1,2,\ldots,n\}$ ∪ {X – A} covers X. But (X – A) covers no part from A, implies, $\{U_{\lambda_i}:i = 1,2,\ldots,n\}$ covers A. So {α-closur U_{λ_i} : i = 1,2,…,n} covers A. Hence, X is α-c-compact space.

2.3 Corollary

If every nowhere dense subset of a topological space (X,τ) is finite, then X is α-c-compact space whenever it is compact.

Proof:

Follows from propositions (1.8) and (2.2).

2.4 Corollary

If every nowhere dense set is finite in a T_3-c-compact space (X,τ), then it is α-c-compact space.

Proof:

Follows from propositions (1.4) and corollary (2.3).

2.5 Remark

The opposite direction of proposition (2.2) may be untrue. For example:

Let N be the set of all natural numbers, and let $\tau = \{\varnothing,\{1\},N\}$ be a topology on N. Then $\{\{1,n\} \mid n \in N\}$ is an α-open cover for N which has no finite subcover. So N is not α-compact space. But N is α-c-compact, since N is the unique α-closed set contains 1.

In the following proposition we put some condition to make the α-c-compact space an α-compact space.

2.6 Proposition

A T_3-α-c-compact space is α-compact.

Proof:

Let X be a T_3-α-c-compact space, if it is not α-compact, then there is an α-open cover for X say $\{U_\lambda:\lambda \in\wedge\}$ which has no finite subcover. Since X is α-c-compact space, then there is a finite subfamily $\{ U_{\lambda_i} : i = 1,2,\ldots,n\}$ such that $\{\alpha$-closure $U_{\lambda_i} : i = 1,2,\ldots,n\}$ covers X. This means, there existS $x \in X$ such that $x \in \alpha$-cl U_{λ_i} and $x \notin U_{\lambda_i}$ for some $i = 1,2,\ldots,n$. Implies $x \in \alpha$-derived U_{λ_i} for some $i = 1,2,\ldots,n$.

Now, since X is T_1-space, then $\{x\}$ is closed set and since $x \notin U_{\lambda_i}$, then $y \notin \{x\}$ for each $y \in U_{\lambda_i}$ and X is regular space, implies for each $y \in U_{\lambda_i}$, there are two open sets V_y and V'_y such that $y \in V_y$ and $\{x\} \subseteq V'_y$ and $V_y, \cap V'_y = \varnothing$. Implies, $\{x\} \subseteq \cup \{V'_y: y \in U_{\lambda_i}\}$ and $U_{\lambda_i} \subseteq \cup\{V_y:y \in U_{\lambda_i}\}$. But $\{x\}$ is compact set, then there is a finite subset of U_{λ_i} say $\{y_1, y_2, \ldots, y_n\}$ such that $\{x\} \subseteq \cup \{ V'_{y_j} : j = 1,2,\ldots,n\}$.

Now, let $V' = \cap \{ V'_{y_j} : j = 1,2,\ldots,n\}$, then V' is an open set contains x. On the other side, let $V = \cup \{V_y : y \in U_{\lambda_i}\}$ implies V is an open set contains U_{λ_i}. So $V \cap V' = \varnothing$. In view of, every open set is α-open, hence, $x \notin \alpha$-derived U_{λ_i} which is a contradiction. thereupon, X is α-compact space.

2.7 Corollary

A T_3-α-c-compact space is compact.

Proof:

In view of, every α-compact space is compact, then proposition (2.6) is applicable. ■

2.8 Remark

In general, α-c-compact space need not be compact as the following example shows:

Let N be the set of all natural numbers and let $\tau = \{U_n \mid u_n = \{1,2,\ldots,n\}; n \in N\} \cup \{\varnothing,N\}$. Then (N,τ) is α-c-compact space, since N is the unique α-closed set contains 1. But N is not compact space.

In corollary (2.4), we discussed the relationship between, c-compact and α-c-compact space, in one side, the other side of this relation we shall descry in the following proposition.

2.9 Proposition

An α-c-compact space is c-compact.

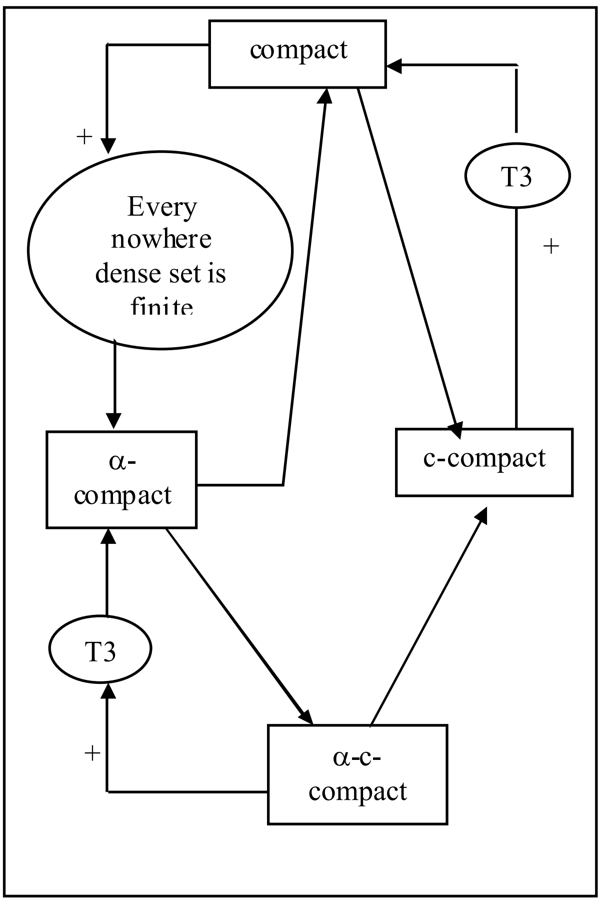
Proof:

Let X be an α-c-compact space. If it is not c-compact space, then there is a closed set A $\subseteq$ X, and a family of open sets in X say $\{U_\lambda:\lambda\in\wedge\}$ covers A. But for each n ∈ N, implies A $\not\subset \cup \{cl\ U_{\lambda_i}, i = 1,2,\ldots,n\}$.

On the other side, clearly A is α-closed subset of an α-c-compact space X and $\{U_\lambda:\lambda\in\wedge\}$ is an α-open cover for A in X, then there exists n ∈ N such that A $\subseteq \cup\{\alpha\text{-cl } U_{\lambda_i} : i = 1,2,\ldots,n\}$. This means, there exists $x \in$ A such that $x \in \alpha$-cl U_{λ_i} and $x \notin$ cl U_{λ_i} for some i = 1,2,…,n. Since $x \notin$ cl U_{λ_i}, implies $x \notin U_{\lambda_i}$ and $x \notin$ derived U_{λ_i}. But $x \in \alpha$-cl U_{λ_i}, then $x \in$ α-derived U_{λ_i}. Since $x \notin$ derived U_{λ_i} then there exists an open set say V such that $x \in$V and $V \setminus \{x\} \cap U_{\lambda_i} = \emptyset$.

In view of, every open set is α-open then V is α-open set implies $x \notin$ α-derived U_{λ_i} which is a contradiction. Therefore, X is c-compact space whenever it is α-c-compact.

The following diagram shows the relationships among the different types of compactness that we studied in this paper.



3. Certain Fundamental Properties of α-c-compact Spaces

In this section, we shall discuss some properties of the new kind of compactness which we introduced in this paper.

In remark (3.1) and proposition (3.3) we shall discuss the heredity property in α-c-compact spaces.

3.1 Remark

α-c-compactness is not a hereditary property. For example:

Let X = N∪ {-1,0} and τ = P(N) ∪ {H⊆X | -1,0∈H∧X – H is finite}.

Clearly: (X,τ) is α-c-compact space, since the complement of each α-closed set which contains (-1) or (0) is finite set.

Now, take N as a subspace of (X,τ). It is clear that the induced topology on N is the discrete topology on N Hence, N is not α-c-compact space.

The above example shows that if Y is an open subspace of an α-c-compact space (X,τ), then Y need not be α-c-compact.

3.2 Remark [4], [6]

i. If Y is an open subset of a topological space X, then every α-open set in Y is an α-open set in X.

ii. If Y is an open, α-closed subspace of an α-compact space X, then Y is α-compact.

3.3 Proposition

If Y is an open and α-closed subspace of an α-c-compact space X, then Y is α-c-compact.

The proof of this proposition will take effect in virtue of remark (3.2). ■

3.4 Definition [8], [9]

A function f:(X,τ) ⟶ (Y,τ′) is said to be "α*-continuous", if and only if the inverse image of every α-open subset of Y is an α-open subset of X.

3.5 Remark [10]

A function f:(X,τ) ⟶ (Y,τ′) is said to be "α*-continuous", if and only if the inverse image of every α-closed subset of Y is an α-closed subset of X.

3.6 Lemma

A function f:(X,τ) ⟶ (Y,τ′) is α*-continuous if and only if α-closure (f^{-1}(B)) ⊆ f^{-1}(α-closure((B)) for each B ⊆ Y.

Proof:

Necessity, let f:(X,τ) ⟶ (Y,τ′) be an α*-continuous function, let B ⊆ Y.

Now, since, B ⊆ α-cl B, then (f^{-1}(B)) ⊆ f^{-1}(α-cl B), implies, α-cl(f^{-1}(B)) ⊆ α-cl(f^{-1}(α-cl B)). In virtue of remark (3.5), f^{-1}(α-cl B) is an α-closed set in X. So α-cl(f^{-1}(α-cl B)) = f^{-1}(α-cl B). Therefore α-cl(f^{-1}(B)) ⊆ f^{-1}(α-cl B).

Sufficiency, suppose α-cl(f^{-1}(B)) ⊆ f^{-1}(α-cl B) for each B ⊆ Y. To prove f is α*-continuous function. We must prove if A ia an α-closed set in Y, then f^{-1}(A) is an α-closed set in X. It is enough to prove that α-cl(f^{-1}(A)) ⊆ f^{-1}(A).

Since A is α-closed set in Y, then α-cl(A) = A and by hypothesis, α-cl(f^{-1}(A)) ⊆ f^{-1}(α-cl (A)) implies,α-cl(f^{-1}(A)) ⊆ f^{-1} (A). So f^{-1}(A) is an α-closed set in X and f is α*-continuous function.

3.7 Prposition

The α*-continuous image of an α-c-compact space is α-c-compact.

Proof:

Let (X,τ) be an α-c-compact space, and f :(X,τ) ⟶ (Y,τ′) be an α*-continuous onto function. To prove (Y,τ′) is α-c-compact space. Let A be an α-closed subset of Y, and {U_λ: λ∈∧} be an α-open cover in Y for A. Since f is α*-continuous, then f^{-1}(A) is an α-closed set in X and { f^{-1}(U_λ):λ∈∧} is a family of α-open sets in X covering f^{-1}(A) and X is α-c-compact space, then there is $\lambda_1, \lambda_2,\ldots,\lambda_n$ such that {α-cl(f^{-1}(U_{λ_i})):i = 1,2,…,n} covers f^{-1}(A), implies {f(α-cl(f^{-1}(U_{λ_i}))): i = 1,2,…,n} covers A. In virtue of lemma (3.6), {f (f^{-1} (α-cl(U_{λ_i}))):i = 1,2,…,n} covers A. Since *f* is onto function,{α-cl(U_{λ_i}):i = 1,2,…,n} covers A. Hence, Y is α-c-compact space.

3.8 Proposition [9], [10]

Every continuous, onto, open function is α*-continuous.

3.9 Corollary

An α-c-compactness is a topological property.

Proof:

Follows from propositions (3.8) and (3.7).

4. Conclusion and Recommandation

We introduced a new type of compactness which is called α-c-compact and discussed the relationships among this type and some types of compactness like, compact, α-compact and c-compact.

Also, we some examples to explain the direction that not hold and we put some condition to make that false direction valid.

In future, we shall study strongly c-compact, semi-α-c-compact, semi-p-compact and semi-p-c-compact.

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