Ibn Al-Haitham Jour. for Pure & Appl. Sci. 34 (2) 2021

Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

The Galerkin-Implicit Methods for Solving Nonlinear Hyperbolic Boundary Value Problem

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Article history: Received 21, April ,2020, Accepted,23,June ,2020, Published in April 2021

Doi: 10.30526/34.2.2618

Abstract

This paper is concerned with finding the approximation solution (APPS) of a certain type of nonlinear hyperbolic boundary value problem (NOLHYBVP). The given BVP is written in its discrete (DI) weak form (WEF), and is proved that it has a unique APPS, which is obtained via the mixed Galerkin finite element method (GFE) with implicit method (MGFEIM) that reduces the problem to solve the Galerkin nonlinear algebraic system (GNAS). In this part, the predictor and the corrector technique (PT and CT) are proved convergent and are used to transform the obtained GNAS to linear (GLAS), then the GLAS is solved using the Cholesky method (ChMe). The stability and the convergence of the method are studied. The results are given by figures and shown the efficiency and accuracy for the method.

Keywords: nonlinear hyperbolic boundary value problem, Galekin finite element method, implicit method, convergence, stability.

1. Introduction

Hyperbolic partial differential equations play a very important role as real life problems in many fields of sciences as in technology, fluid dynamics, optics, science and many others. In the past few decades, there have been many researchers interested in their study to solve boundary value problems in general and in particular NLHBVE. Many researchers have used different methods to solve the NLHBVE, Smiley studied in 1987, was used Eigen function methods to solve problems of nonlinear hyperbolic value at resonance [1]. In 1989, Chi, Wiener, and Shah used in the exponential growth of solutions of nonlinear hyperbolic equations [2], while in 2001 Minamoto used the existence and demonstration of the uniqueness of solutions [3]. In 2004, Krylovas, and Čiegis, used the numerical asymptotic averaging for weakly nonlinear hyperbolic waves [4].

In 2018, Ashyralyev and Agirseven solved NLHBVE with a time delay [5].

The specific element method has been studied by several researchers interested in this field, for example, in 2010 Bangerth and Rannacher touched on Galerkin's specific adaptation techniques for wave equation [6]. Whereas, in 2017, Al-Haq and Muhammad discussed numerical methods to solve LHYBVP by difference method and the method of the specified elements [7].

In this paper, we are concerned the study of the APPS of the NOLHYBVP. The given BVP is written in its WEF, and in its discrete equation (DI) type. It is proved to have unique APPS. The APPS is obtained via the MGFEIM. The problem then reduces to solve the GNAS, then the PT and CT are proved convergent and are used to transform the GNAS to a GLAS. This GLAS is solved by using the ChMe. The stability and the convergence of the method are studied. A computer program is codding to find the numerical solution for the problem. The results are given by figures, and are shown the efficiency and accuracy for the method which is highly considered in this work.

2. Description of The NOLHYBVP

Let $I = [0, \mathrm{T}]$, with $0 < T < \infty$, $\psi \subset \mathbb{R}^2$ be a bounded and open region with smooth boundary $\partial\psi$, $\varphi = \psi \times I$, $\Sigma = \partial\psi \times I$, then the NLHYPVP is given by:

$w_{tt} - \Delta w + w = h(\vec{x}, t, w)$, in φ
(1)

$w(\vec{x}, 0) = w^0(\vec{x})$, in ψ
(2)

$w_t(\vec{x}, 0) = w^1(\vec{x})$, in ψ
(3)

$w(\vec{x}, t) = 0$, on Σ
(4) where $w = w(\vec{x}, t) \in H_0^2(\psi)$,, $\Delta w = \sum_{i=1}^{2} \frac{\partial^2 w}{\partial x_i^2}$ and $h \in L^2(\psi)$ is a given function.

Now, let $V = H_0^1(\psi)$={ η:$\eta \in H^1(\psi)$, $\eta = 0$ on $\partial\psi$}, $w_t = p$, then the WEFM of (1-4) is:

$\langle w_{tt}, \eta \rangle + (\nabla w, \nabla \eta) + (w, \eta) = (h(w), \eta)$, $\forall \eta \in V$ are on I,
(5)

$(w(0), \eta) = (w^0, \eta)$ in ψ, $w^0 \in V$
(6)

$(p(0), \eta) = (w^1, \eta)$ in ψ, $w^1 \in L^2(\varphi)$,
(7)

Definition (1),[8]: A point $s^* \in D \subset \mathbb{R}^2$ is called a fixed point of the function $y : D \longrightarrow \mathbb{R}^2$,
if $y(s^*) = s^*$.

Definition 2,[8]: A function $y: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is called contractive on D if for each $d_1, d_2 \in D$:

$\|y(d_2) - y(d_1)\| \leq a\|d_2 - d_1\|$, where $a \in (0,1)$.

Theorem (3),[8]: A contractive function y on a complete normed space D has a unique fixed point s^* in D.

Theorem (4),[9]: Let $\{v_n\}$ be a bounded sequence in the space in $L^\infty(\psi)$. Then, there exists a subsequence $\{n'\}$ and a function $v_0 \in L^\infty(\psi)$ such that, in $L^\infty(\psi)$ then $v_{n'} \longrightarrow v_0$.

Assumptions 2.1:

(i) Let κ_1 and κ_2 be two positive constants such that the following are satisfied:

a) $|(\nabla\mu_1, \nabla\mu_2)| \leq \kappa_1 \| \nabla\mu_1 \|_1 \| \nabla\mu_2 \|_1$, $\forall$ μ_1 , $\mu_2 \in V$

b) $(\nabla\mu, \nabla\mu) \geq \kappa_2 \| \nabla\mu \|_1^2$, $\forall$ $\mu \in V$

(ii) The function h is defined on $\varphi \times \mathbb{R}$, continuous with respect to w_j^n satisfies the following:

a) $|h(\vec{x}, t, w)| \leq \beta(\vec{x}, t) + \delta |w|$ where $\delta > 0$, $w \in \varphi$ and $\beta \in L^2(\varphi)$.

b) $|h(\vec{x}, t, w_1) - h(\vec{x}, t, w_2)| \leq L|w_1 - w_2|$, where L is a Lipchitz constant and $w_1, w_2 \in \mathbb{R}$.

3. Discretization of the Continuous Equation (COE):

The WEF of ((5)- (7)) is discretized by using the GFEME , let φ be divided into sub regions $\varphi_{ij} = \psi_i^n \times I_j^n$, let $\{\psi_i^n\}_{i=1}^{N(n)}$ be a triangulation of $\bar{\psi}$ and $\{I_j^n\}_{j=0}$ be a subdivision of the interval $\bar{I}$ into Y(n) intervals, then $I_j = I_j^n := [t_j^n, t_{j+1}^n]$ has the same length $\Delta t = \frac{T}{Y}$, also, let $V_n \subset V = H_0^1(\psi)$ be the space of piecewise affine functions in ψ.

Now, the discrete equations (DES), where $\forall\, \eta \in V_n$ are written as follows:

$\langle p_{j+1}^n - p_j^n, \eta \rangle + \Delta t\, (\nabla w_{j+1}^n, \nabla \eta) + \Delta t\, (w_{j+1}^n, \eta) = \Delta t\, (h(w_{j+1}^n), \eta)$

(8)

$w_{j+1}^n - w_j^n = \Delta t\, p_{j+1}^n$

(9) $(w(0), \eta) = (w^0, \eta)$ in ψ

(10)

$(p(0), \eta) = (w^1, \eta)$ in ψ

(11)

where, $w^0 \in V$, $w^1 \in L^2(\psi)$, and $w_j^n = w^n(x, t_j^n), p_j^n = p^n(\vec{x}, t_j^n) \in V_n$, $\forall$ $j = 0,1, \dots, Y - 1$.

4. The APPS of the NLHYBVP:

To find the APPS $\overline{w}^n = (w_0^n, w_1^n, \dots, w_Y^n)$for the DES (8-11), the MGFEIM is used through the following steps:

(1)Let $\{ \eta_i : i = 1,2, \dots. N, with\, \eta_i(\vec{x}) = 0 \,, on\ \partial\psi \}$ be a basis of V_n, and by using the GFEME, let $\overline{w}^n(\vec{x}, t_j^n)$ (with $\overline{w}_t^n(\vec{x}, t_j^n) = \bar{p}(\vec{x}, t_j^n)$)be an APPS of (8-11) such that $\overline{w}^n(\vec{x}, t_j^n) = \sum_{k=1}^{N} r_k^j \eta_i$ and $\bar{p}^n(\vec{x}, t_j^n) = \sum_{k=1}^{N} u_k^j \eta_i$ $\forall\, \eta_i \in V_n$,

where $r_k^j = r_k(t_j^n)$,and $u_k^j = u_k(t_j^n)$ are unknown constants $\forall j = 0,1, \dots, Y - 1$.

(2) Using the APPs in (8-11) to get , $\forall j = 0,1, \dots, Y - 1$:

$$(M + (\Delta t)^2 Q) R^{j+1} = M R^j + (\Delta t)\, M U^j + (\Delta t)^2\, \vec{L}\left(t_j^n, \vec{v}^T R^{j+1}\right) \tag{12}$$

$$U^{j+1} = \frac{1}{\Delta t}(R^{j+1} - R^j) \tag{13}$$

$$M R^0 = s^0 \tag{14}$$

$$M U^0 = s^1 \tag{15}$$

where, $M = (m_{ik})_{N\times N}$, $m_{ik} = (\eta_k, \eta_i)$, $Q = (q_{ik})_{N\times N}$, $q_{ik} = (\nabla\eta_k, \nabla\eta_i)$, $\vec{L} = (L_i)_{N\times 1}$, $L_i = (h(\vec{v}^T R^{j+1}), \eta_i)$, $R_{N\times 1}^j = (r_1^j, r_2^j, \dots, r_N^j)^T$,$U_{N\times 1}^j = (u_1^j, u_2^j, \dots, u_N^j)^T$, $s^0 = (s_i^0)_{N\times 1}$,

$s_i^0 = (w^0, \eta_i)$, $s^1 = (s_i^1)_{N\times 1}$ and $s_i^1 = (w^1, \eta_i)$, for each $i, k = 1,2, \dots, N$.

(3) System (12-15), is GNAS and has a unique solution. To solve it, we find at first R^0 and U^0 from solving (14) and (15) respectively, then, the PT and the CT are utilized to solve (12) for each $j(\ j=0,1,\ldots,Y-1)$ as follows:
In the PT we suppose $R^{j+1}=R^j$ in the components of $\vec{L}$ in the R.H.S of (12), then it turn to a GLAS, which is solved to get the predictor solution R^{j+1}, then in the CT we resolve (12) with setting $\bar{R}^{j+1}=R^{j+1}$ (in the components of $\vec{L}$ of the R.H.S of it) to get the corrector solution R^{j+1}, finally substituting R^{j+1} in (13) to get U^{j+1}, we can repeat this procedure if we want more than one time. This reputation can be expressed as follows:

$$(w_{j+1}^{(l+1)},\eta_i)+\left(\Delta\mathrm{t}\right)^2(\nabla w_{j+1}^{(l+1)},\nabla\eta_i\Big)+\left(\Delta\mathrm{t}\right)^2\left(w_{j+1}^{(l+1)},\eta_i\right)=(w_j^n,\eta_i)+\Delta\mathrm{t}\,(p_j^n,\eta_i)$$
$$+(\Delta\mathrm{t})^2 h\left(t_j^n,w_{j+1}^{(l)}\right),\eta_i) \tag{16}$$

$$p_{j+1}^{(l+1)}=\frac{(w_{j+1}^{(l+1)}-w_j^n)}{\Delta\mathrm{t}} \tag{17}$$

Equation (17) tells us the iterative method depending on just $w_{j+1}^{(l+1)}$. Thus, equation (16) is reformulated as $w^{(l+1)}=\delta(w^{(l+1)})$, where l is the number of the iterations. And this led us to the following theorem.

Theorem (5): For any fixed point, the DES (8)-(11), and for Δ sufficiently small, has a unique solution $w^n=(w_0^n,w_1^n,\ldots\ldots,w_N^n)$ and the sequence of the corrector solutions converges on $\mathbb{R}$.

proof: Let $w^{(l+1)}=(w_0^{(l+1)},w_1^{(l+1)},\ldots\ldots,w_N^{(l+1)})$ and
$\overline{w}^{(l+1)}=(\overline{w}_0^{(l+1)},\overline{w}_1^{(l+1)},\ldots\ldots,\overline{w}_N^{(l+1)})$ where $w^{(l+1)}$ and $\overline{w}^{(l+1)}$ are solutions of equation (16). This means,

$$(w_{j+1}^{(l+1)},\eta_i)+(\Delta\mathrm{t})^2(\nabla w_{j+1}^{(l+1)},\nabla\,\eta_i)+(\Delta\mathrm{t})^2\,(w_{j+1}^{(l+1)},\eta_i)=(w_j^n,\eta_i)+\Delta\mathrm{t}\,(p_j^n,\eta_i)$$
$$+(\Delta\mathrm{t})^2\,(h\left(t_j^n,w_{j+1}^{(l)}\right),\eta_i\,) \tag{18}$$

and

$$(\overline{w}_{j+1}^{(l+1)},\eta_i)+(\Delta\mathrm{t})^2(\nabla\,\overline{w}_{j+1}^{(l+1)},\nabla\,\eta_i)+(\Delta\mathrm{t})^2\,(\overline{w}_{j+1}^{(l+1)},\eta_i)=(w_j^n,\eta_i)+\Delta\mathrm{t}\,(p_j^n,\eta_i)$$
$$+(\Delta\mathrm{t})^2(h\left(t_j^n,\overline{w}_{j+1}^{(l)}\right),\eta_i\,) \tag{19}$$

subtracting (19) from (18) then setting $\eta_i=(\overline{w}_{j+1}^{(l+1)}-w_{j+1}^{(l+1)})$ in the obtained equation , we get that

$$(\overline{w}_{j+1}^{(l+1)}-w_{j+1}^{(l+1)},\,\overline{w}_{j+1}^{(l+1)}-w_{j+1}^{(l+1)})+(\Delta\mathrm{t})^2(\nabla\,\overline{w}_{j+1}^{(l+1)}-\nabla w_{j+1}^{(l+1)},\,\nabla\,\overline{w}_{j+1}^{(l+1)}-\nabla w_{j+1}^{(l+1)})+$$
$$(\Delta\mathrm{t})^2(\overline{w}_{j+1}^{(l+1)}-w_{j+1}^{(l+1)},\,\overline{w}_{j+1}^{(l+1)}-w_{j+1}^{(l+1)})=(\Delta\mathrm{t})^2(h\left(\overline{w}_{j+1}^{(l)}\right)-h\left(w_{j+1}^{(l)}\right),\,\overline{w}_{j+1}^{(l+1)}-w_{j+1}^{(l+1)})$$
(20)

From Assumptions 2.1 (ib) the 2^{nd} and 3^{rd} terms in the L.H.S of equation (20) are positive, and applying Assumption 2.1 (iib) on h in R.H.S of equation (20), and by using the Cauchy Schwarz inequality on this side, we get

$$\left\|\delta\left(\overline{w}_{j+1}^{(l)}\right)-\delta(w_{j+1}^{(l)})\right\|_0=\left\|\overline{w}_{j+1}^{(l+1)}-w_{j+1}^{(l+1)}\right\|_0\le\lambda\left\|\overline{w}_{j+1}^{(l)}-w_{j+1}^{(l)}\right\|_0 \tag{21}$$

where $\lambda=(\Delta\mathrm{t})^2L<1$, for sufficiently small Δt.
which implies that δ is contractive, also since $\{w^{(l)}\}\in\mathbb{R}\;\forall\,l,$ that $\delta\left(w^{(l+1)}\right)=w^{(l+1)}\in\mathbb{R}\;\forall\,l,\ i.e\ \delta(w)\in\mathbb{R}$, hence ,by theorem 3 the sequence $\{w^{(l)}\}$ converges to a point in $\mathbb{R}$.

5. Stability:

Lemma (6): If Δ is sufficiently small, then $\forall\, j = 0,1,\dots,Y$

$\|w_j^n\|_1^2 \le \bar{d}$, $\|p_j^n\|_0^2 \le \bar{d}$, $\sum_{j=0}^{Y-1} \|w_{j+1}^n - w_j^n\|_1^2 \le \bar{d}$, and $\sum_{j=0}^{Y-1} \|p_{j+1}^n - p_j^n\|_0^2 \le \bar{d}$

where $\bar{d}$ refers to a various constants.

proof: Let $\eta = p_{j+1}^n$ substituting in equation (8), and rewriting the first term in the L.H.S of the obtained equation, we get

$$\|p_{j+1}^n\|_0^2 - \|p_j^n\|_0^2 + \|p_{j+1}^n - p_j^n\|_0^2 + \Delta t\,(\nabla w_{j+1}^n, \nabla p_{j+1}^n) + \Delta t\,(w_{j+1}^n, p_{j+1}^n) = \Delta t\,(h(w_{j+1}^n), p_{j+1}^n) \quad (22)$$

Since,

$$\Delta t\,[(\nabla w_{j+1}^n, \nabla p_{j+1}^n) + (w_{j+1}^n, p_{j+1}^n)] = \frac{1}{2}[(\nabla w_{j+1}^n - \nabla w_j^n, \nabla w_{j+1}^n - \nabla w_j^n) + w_{j+1}^n - w_j^n, w_{j+1}^n - w_j^n) + (\nabla w_{j+1}^n, \nabla w_{j+1}^n) + (w_{j+1}^n, w_{j+1}^n) - (\nabla w_j^n, \nabla w_j^n) - (w_j^n, w_j^n)]$$

By substituting above equality in the L.H.S in equation (22), summing both sides of the obtained equality, for $j = 0\ to\ j = l-1$, then set c = max $(1, \frac{k_2}{2})$, we get

$$c\|p_l^n\|_0^2 + c\sum_{j=0}^{l-1}\|p_{j+1}^n - p_j^n\|_0^2 + c\|w_l^n\|_1^2 + c\sum_{j=0}^{l-1}\|w_{j+1}^n - w_j^n\|_1^2 \le \|p_0^n\|_0^2 + \frac{k_2}{2}\|w_l^n\|_1^2 + \sum_{j=0}^{l-1}\Delta t\,(h(w_{j+1}^n), p_{j+1}^n) \quad (23)$$

Now, using the assumptions on h and then by the Cauchy Schwarz inequality, to get

$$|(h(w_{j+1}^n), p_{j+1}^n)| \le \|\beta_j\|_0^2 + \delta\|w_{j+1}^n\|_1^2 + \bar{\delta}\|p_{j+1}^n\|_0^2,\ \bar{\delta} = \delta + 1 \quad (24)$$

since $\|w_{j+1}^n\|_1^2 = 2\|w_{j+1}^n - w_j^n\|_1^2 + 2\|w_j^n\|_1^2$ (25)

and $\|p_{j+1}^n\|_0^2 = 2\|p_{j+1}^n - p_j^n\|_0^2 + 2\|p_j^n\|_0^2$ (26)

Substituting (25) and (26) in inequality (23), and assume that $d = \max(2\delta, 2\bar{\delta})$, to get

$$c\|p_l^n\|_0^2 + (c - d\Delta t)\sum_{j=0}^{l-1}\|p_{j+1}^n - p_j^n\|_0^2 + c\|w_l^n\|_1^2 + (c - d\Delta t)\sum_{j=0}^{l-1}\|w_{j+1}^n - w_j^n\|_1^2 \le \|p_0^n\|_0^2 + \frac{k_2}{2}\|w_l^n\|_1^2 + \|\beta\|_Q^2 + d(\Delta t)\sum_{j=0}^{l-1}\|w_j^n\|_1^2 + d(\Delta t)\sum_{j=0}^{l-1}\|p_j^n\|_0^2. \quad (27)$$

Now, let $\Delta t < c/d$ then the 2^{nd} and 4^{th} terms in the R.H.S of (27) are positives, by using the discrete Gronwall's (DGs) inequality [10], one obtains

$c(\|p_l^n\|_0^2 + \|w_l^n\|_1^2) \le ae^{\sum_{j=0}^{l-1} d(\Delta t)} = ae^{ld(\Delta t)} \le b$,

which gives that

$\|w_l^n\|_1^2 \le d_1 = \frac{b}{c}$, and $\|p_l^n\|_0^2 \le d_1$, for any arbitrary index l.

Hence, $\|w_j^n\|_1^2 \le d_1$ and $\|p_j^n\|_0^2 \le d_1$, for each $j = 0,1,\dots\dots, Y-1$.

Therefore

$$(\Delta t)d\sum_{j=0}^{Y-1}\|w_j^n\|_1^2 + (\Delta t)d\sum_{j=0}^{Y-1}\|p_j^n\|_1^2 \le 2d_1\, d\,\Delta t\, Y = 2cT = \bar{d}.$$

We back to (27) substituting $l = Y$, the 1st and the 3rd term in the L.H.S are positives, then we use the above results in the R.H.S. of it , keeping in mind the first three terms in this side that are bounded (from the above steps), to obtain

$$\sum_{j=0}^{Y-1}\|w_{j+1}^n - w_j^n\|_1^2 \le \bar{d} \quad (28a)$$

$$\sum_{j=0}^{Y-1}\|p_{j+1}^n - p_j^n\|_0^2 \le \bar{d} \quad (28b)$$

6. Convergence:

The following definitions for the functions "almost everywhere on $\bar{I}$ " are useful in the proof of next theorem, so let

$w_{-}^{n}(t) := w_j^n,\ t \in I_j^n,\ \forall j = 0,1,\dots,Y,$

$w_{+}^{n}(t) := w_{j+1}^n,\ t \in I_j^n, \forall j = 0,1,\dots,Y-1,$

$p_{+}^{n}(t) := p_{j+1}^n,\ t \in I_j^n, \forall j = 0,1,\dots,Y-1,$

$p_{-}^{n}(t_j^n) := p_j^n, t_j^n \in I_j^n,\ \forall j = 0,1,\dots,Y,$

Also, Let $w_{\wedge}^{n}(t) := w_j^n$ be an affine function on each I_j^n, $\forall, j = 0,1,\dots,Y$, and $p_{\wedge}^{n}(t) := p_j^n$, be an affine function on each I_j^n, $\forall, j = 0,1,\dots,Y$.

Theorem (7): The discrete solutions $w_{-}^{n}(t), w_{+}^{n}(t)$, and $w_{\wedge}^{n}(t)$ are converges strongly in $L^2(\varphi)$, where $n \longrightarrow \infty$.

proof: we start with using lemma (6) we have for any $j = 0,1,\dots,Y$ that

$\|w_j^n\|_1^2 \le \bar{d}$ and $\|p_j^n\|_0^2 \le \bar{d}$, then $\|w_{-}^{n}\|_{L^2(I,V)}^2, \|w_{+}^{n}\|_{L^2(I,V)}^2, \|w_{\wedge}^{n}\|_{L^2(I,V)}^2, \|p_{-}^{n}\|_{L^2(\varphi)}^2, \|p_{+}^{n}\|_{L^2(\varphi)}^2$, and $\|p_{\wedge}^{n}\|_{L^2(\varphi)}^2$ are bounded.

From (28a), we have

$\Delta t \sum_{j=0}^{Y-1} \| w_{j+1}^n - w_j^n \|_0^2 \le \Delta t \bar{d} \longrightarrow 0$, as $\Delta t \longrightarrow 0$,

gives

$\bar{w}_{+}^{n} \longrightarrow \bar{w}_{-}^{n}$ strongly (ST) in $L^2(I,V)$ and then in $L^2(\varphi)$. (29a)

by the same way from (28b), we get that ,

$p_{+}^{n} \longrightarrow p_{-}^{n}$ ST in $L^2(\varphi)$ (29b)

Then by theorem 3, there exist subsequences of $\{w_{-}^{n}\}$, $\{w_{+}^{n}\}$, $\{w_{\wedge}^{n}\}$), and of ($\{p_{-}^{n}\}$, $\{p_{+}^{n}\}$, $\{p_{\wedge}^{n}\}$,) use again the same notations which converge weakly to some w in $L^2(I,V)$, to some p in $L^2(\varphi)$, i.e.

$$w_{-}^{n} \longrightarrow w, w_{+}^{n} \longrightarrow w, w_{\wedge}^{n} \longrightarrow w \text{ weakly in } L^2(I,V)$$

$$p_{-}^{n} \longrightarrow p, p_{+}^{n} \longrightarrow p, p_{\wedge}^{n} \longrightarrow p \text{ weakly in } L^2(\varphi),$$

N this point the first compactness theorem[9] is used , to get that $w_{\wedge}^{n} \longrightarrow w$ ST in $L^2(\varphi)$,then $w_{+}^{n} \longrightarrow w$ and $w_{-}^{n} \longrightarrow w$ ST in $L^2(\varphi)$.

Now, let $\{V_n\}_{n=1}^{\infty}$ be a sequence of subspaces of V, where V_n is as defined above. Then by using the Galerkin approach, for each $\eta \in V$, there exists a sequence $\{\eta_n\}$, with $\eta_n \in V_n$ for each n, such that $\eta_n \longrightarrow \eta$ ST in $L^2(\varphi)$.

Consider that $\xi(t) \in C^2[0,T]$, for which $\xi(T) = \xi'(T) = 0$ and $\xi(0) = \xi'(0) \neq 0$, let $\xi^n(t)$ continuous piecewise(CP) interpolation of $\xi(t)$ with respect to I_j^n, and let $\zeta = \eta\, \xi(t)$, with $\zeta^n = \eta_n \xi^n(t)$, with

$\zeta_{-}^{n} := \eta_n \xi_{-}^{n}(t), t \in I_j^n,\ j = 0,1,\dots,Y-1, \eta_n \in V_n,$

$\zeta_{+}^{n} := \eta_n \xi_{+}^{n}(t),\ t \in I_j^n,\ \ j = 0,1,\dots,Y-1, \eta_n \in V_n,$

$\zeta_{\wedge}^{n} := \eta_n \xi^n(t),\ t \in I,\ \ \eta_n \in V_n,$

Setting $\eta = \zeta_{j+1}^n$ in equation (8), and summing both sides of the obtained equation for $j = 0$, to $j = Y-1$, then using discrete integrating by parts (DIBP) for the 1st term in the L.H.S., once can get that

$-\int_0^T (p_-^n ,(\zeta_\wedge^n)')\,dt \ + \int_0^T [(\nabla w_+^n ,\nabla \zeta_+^n) + (w_+^n ,\zeta_+^n)]\,dt = \int_0^T (h(t_-^n ,w_+^n),\zeta_+^n)\,dt +$
$(p_0^n ,\eta_n)\,\xi(0)$
(30)
On the other hand, from (9), once has
$((w_\wedge^n)',\eta_n)(\xi^n)' = (p_+^n ,\eta_n)(\xi^n)'$
Integrating both sides on $[0,T]$, then using DEBP for the 1st term in the L.H.S., to obtain
$-\int_0^T (w_+^n ,\eta_n)\,(\xi^n(t))''dt \ = \int_0^T (p_+^n \ ,\eta_n)(\xi^n)'dt + (w_0^n ,\eta_n)(\xi^n(0))'$
(31)
Now, since
$\xi^n(t) \longrightarrow \xi(t)$ in $C(I) \subset L^2(I)$, $\eta_n \longrightarrow \eta$ ST in $L^2(I,V)$ and in $L^2(\psi)$, then, we have
$\zeta_+^n = \eta_n \xi_+^n \longrightarrow \eta\,\xi = \zeta$ ST in $L^2(I,V)$ and in $L^2(\varphi)$, $\eta_n \xi^n(0) \longrightarrow \eta\,\xi(0)$ ST in $L^2(\varphi)$,
$(\zeta_\wedge^n)' = \eta_n \xi'^n \longrightarrow \eta\,\xi' = \eta\zeta'$ ST in $L^2(I,V)$.
And since , $t_-^n \longrightarrow t$ ST in $L^\infty(I)$, w_+^n, w_-^n, $w_\wedge^n \longrightarrow w$ ST in $L^2(\varphi)$, $w_0^n \longrightarrow w^0$ ST in V and $p_0^n \longrightarrow w^1$ ST in $L^2(\psi)$.
Now, from assumptions h, and the above convergences, one can passage to the limit in (30) and in (31), to obtain

$$-\int_0^T (p,\eta)\,\xi' dt + \int_0^T [(\nabla w,\nabla\eta) + (w,\eta)]\,\xi\,dt = \int_0^T (h(t,w),\eta)\,\xi\,dt + (w^1, \eta)\,\xi(0) \quad (32)$$

and

$$-\int_0^T (w,\eta)\xi''(t)\,dt = \int_0^T (p \ ,\eta)\xi'(t)dt + (w^0,\eta)\xi'(0) \quad (33)$$

The following cases appear:
Case (1) : Consider $\xi(t) \in C^2[0,T]$, such that $\xi(T) = \xi'(T) = \xi(0) = \xi'(0) = 0$, by setting $\xi'(0) = 0$ in equation (32) and $\xi(0) = 0$ in (33), then we use IBP for the 1st term of each one of the obtained equation, one gets respectively

$$\int_0^T (w_t,\eta)\xi'(t)\,dt = \int_0^T (p,\eta)\xi'(t)dt \implies w_t = p\,,$$

$$\int_0^T (w_{tt},\eta)\,\xi\,dt + \int_0^T [(\nabla w,\nabla\eta) + (w,v)]\,\xi\,dt = \int_0^T (h(t,w),\eta)\xi\,dt, \quad (34)$$

Thus

$$(w_{tt},\eta) + (\nabla \mathrm{w},\nabla\eta) + (w,\eta) = (h(t,w),\eta)\,, \eta \in V \text{ a.e. on } I.$$

Case (2) : Consider $\xi(t) \in D[0,T]$, $\xi(0) \neq 0$, $\xi(T) = 0$ and use IBP the first term in the L.H.S of (34), once get that
$-\int_0^T (w_t,\eta)\,\xi' dt + \int_0^T [(\nabla w,\nabla\eta) + (w,\eta)]\,\xi\,dt = \int_0^T (h(t,w),\eta)\xi\,dt + w_t(0), \ \eta)\xi(0)$
(35)
Setting $p = w_t$ in (32), subtracting the resulting equation from (35), to get
$(w_t(0),\eta)\xi(0) = (w^1, \eta)\xi(0) \implies (w_t(0),\eta) = (w^1, \eta)$, for each η
then $w_t(0) = w^1(0)$.

Case (3): Consider $\xi(t) \in D[0,T]$, with $\xi'(0) \neq 0$, $\xi(0) = 0$, and $\xi(T) = \xi'(T) = 0$. Using twice the IBP for the 1st term in the L.H.S. of (34), to obtain

$$\int_0^T (w,\eta)\xi'' dt + \int_0^T [(\nabla w,\nabla\eta) + (w,\eta)]\xi\,dt = \int_0^T (h(t,w),\eta)\xi\,dt - (w(0),\eta)\xi'(0) \quad (36)$$

Rewritten (33), in the following form

$$-\int_0^T (p,\eta)\xi'(t)\,dt = \int_0^T (w,\eta)\xi''(t)dt + (w^0,\eta)\xi'(0) \quad (37)$$

Substituting (37) in (32), and using $\xi(0) = 0$, then subtracting the resulting equation from (36) to get
$(w(0)\,,\eta)\xi'(0) = (w^0,\ \eta)\xi'^{(0)} \Longrightarrow (w(0)\,,\eta) = (w^0,\ \eta)$ for each η , then $w(0) = w^0(0)$,
Thus limit point w is a solution to the WEF for the COE.

7. Cholesky factorization

Cholesky method is used using to solve GLAS with conditions that the coefficient matrix A must be a symmetric and positive definite. In this method the matrix A can be factorized into the product of an Upper triangular matrix L and Lower triangular matrix L^T [11], and L calculates as follows:

$$for\ i = 1,2,\dots,n \quad then \quad l_{ii} = (a_{ii} - \sum_{k=1}^{i-1} l_{ki}^2\,)^{\frac{1}{2}}$$

$$for\ j = i+1,\dots,n. \quad then \quad l_{ij} = \left(a_{ij} - \sum_{k=1}^{i-1} l_{ki}\, l_{kj}\right) / \, l_{ii}$$

8. Numerical Examples:

The problems in the following examples are coded by Mat lap soft.

Example 1: Consider the following NLHBVP:

$w_{tt} - \Delta\, w + w = h(\vec{x},t,w),\ \vec{x} = (x,y)\,,\varphi = \psi \times I\,,\psi = (0,1)\times(0,1), I = [0,1]$
$w(\vec{x},0) = xy(1-x)(1-y)$, in ψ
$w_t(\vec{x},0) = w^1\,(\vec{x})$, in ψ
$w(\vec{x},t) = 0$, on $\sum = \partial\psi \times I$
where $h(\vec{x},t,w) = \frac{1}{2}(xy - xy^2 - yx^2 + x^2y^2)\sqrt{cos^2}\,[1 - 2\sin(xy - xy^2 - yx^2 + x^2y^2)\sqrt{cost}\,] + 2(y + x - y^2 - x^2)\sqrt{cost} + (xy^2 - yx^2 - xy - x^2y^2)\, sin^2t/4\sqrt[3]{\cos(t)} + w\ sinw$
and the exact solution (EXS)of the problem is $w(\vec{x},t) = xy(1-x)(1-y)\sqrt{\cos(t)}$.

The MGFEIM is utilized to solve this problem with $= 9$, $Y = 20$ and $T = 1$, the results are shown in figure 1. (a) the APPS , and figure 1.(B) shown the EXS at $\hat{t} = 0.5$.

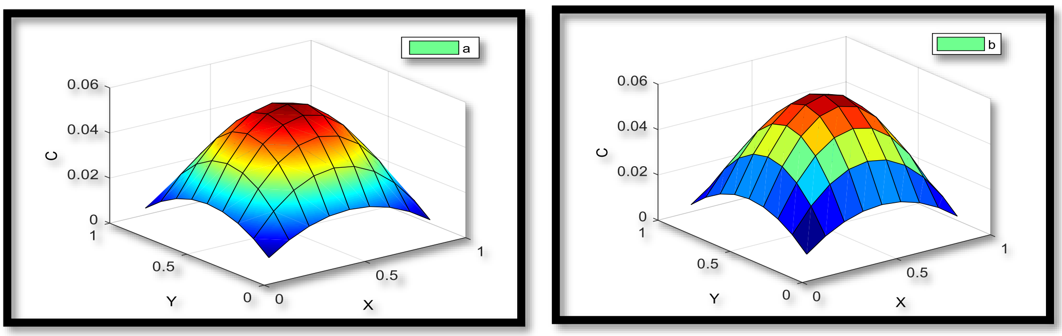

Figure1. (a) shows the APS and (b) shows the EXS

Example 2: Consider the following NLHYBVP :
$w_{tt} - \Delta\, w + w = h(\vec{x},t,w)$ where $\vec{x} = (x,y)$
$w(\vec{x},0) = (x-1)(1-y)\sin(xy)$ in ψ

$w_t\,(\vec{x},0) \;=\; w^1\,(\vec{x})$, in $\;\psi$

$w(\vec{x},t) = 0$, on $\;\sum = \partial\psi \;\times I$

$h(\vec{x},t,w) = 2(y - x - y^2 + x^2)\sqrt{e^{t^2}}\,\cos(xy) + (1 - x - y + xy)\sqrt{e^{t^2}}\,\sin(\mathrm{xy})$

$[x^2 + y^2 - \sin\left((1 - x - y + xy)\sqrt{e^{t^2}}\,\sin(\mathrm{xy})\right)] + w\,\sin(w).$

and the EXS is $w(\vec{x},t) = \;(x-1)(1-y)\sqrt{e^{t^2}}\,\sin(xy)$.

The MGFEIM is utilized to solve this problem with $= 9$, $\;Y = 20$ and $T = 1$, the results are shown in **Figure 2**. **(a)** the APPS , and **Figure 2.(b)** shown the EXS at $\hat{t} = 0.5$.

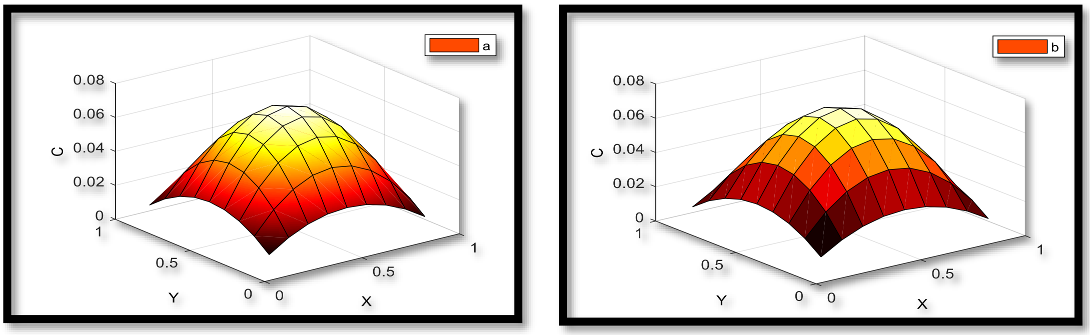

Figure2. (a) shows the APS and (b) shows the EXS

9. Conclusions

The MGFEIM is used successfully to solve the DI of the WEF of a certain type of NOLHYBVP. The existence theorem of a unique convergent APP is proved. The convergent of the PT and CT which are used to solve the GNAS that is obtained from applying the MGFEIM, is proved and the ChMe which is used inside these technique is highly efficient for solving large GAS. The DI of the WEF is proved itis stable and convergent. The results are given by figures and show the efficiency and accuracy for the method.

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