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Study of Fuzzy σ– Ring and Some Related Concepts

Ibrahim S. Ahmed
Mathematical Department /College of Computer Science and Mathematics /Tikrit University
ibrahim1992@tu.edu.iq

Hassan H. Ebrahim
Mathematical Department /College of Computer Science and Mathematics /Tikrit University
hassan1962pl@tu.edu.iq

Ali Al-Fayadh
Department of Mathematics and Computer Applications / College of Science / Al – Nahrain University
aalfayadh@yahoo.com

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Abstract

This paper introduces the concept of fuzzy σ– ring as a generalization of fuzzy σ −algebra and basic properties; examples of this concept have been given. As the first result, it has been proved that every fuzzy σ– algebra over a fuzzy set $\mathcal{X}^*$ is a fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$ and construct their converse by example. Furthermore, the fuzzy ring concept has been studied to generalize fuzzy algebra and its relation. Investigating that the concept of fuzzy σ– ring is a stronger form of a fuzzy ring that is every fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$ and construct their converse by example. In addition, the idea of the smallest, as an important property in the study of real analysis, is studied as well. Finally, the main goal of this paper is to study these concepts and give basic properties, examples, characterizations and relationships between them.

Keywords: σ −algebra, σ– ring, fuzzy σ −algebra, fuzzy algebra, measure.

1. Introduction

The generalized measure theory, which is the subject of this thesis emerged from the well-established classical measure theory by the process of generalization. As is well known, classical measures are nonnegative real-valued set functions, each defined on a specific class of subsets of a given universal set, that satisfies certain axiomatic requirements. One of these requirements, crucial to classical measures, is known as the requirement of additively. Measure theory plays a vital role in mathematics, particularly in probability theory's foundation. The theory of measure has been extensively studied and is used in modeling the physical world. The notion of σ–field is essential in measure theory and probability theory. In 2019 Ahmed and Ebrahim [1] studied the concept of σ–field and discussed many details about some generalizations of this concept. They proved some important results in measure theory. Many authors were interested in studying σ–field and σ– ring, for example see [2-6]. Zadeh in 1965 [7] first introduced the concept of the fuzzy set where $\mathcal{X}$ is a nonempty set, then a fuzzy set F in $\mathcal{X}$ was defined as a set of ordered pairs $\{(\omega, \upsilon_F(\omega)) : \omega \in \mathcal{X}\}$ where $\upsilon_F : \mathcal{X} \to [0, 1]$ was a function such that for every $\omega \in \mathcal{X}$, $\upsilon_F(\omega)$ represented the degree of membership of ω in F. Brown [8] and Wang [9] studied some types of fuzzy sets such as fuzzy power set, empty fuzzy set, universal fuzzy set, the complement of a fuzzy set, the union of two fuzzy sets and the intersection of two fuzzy sets. Ahmed et al [10] first introduced the concept of fuzzy $\sigma-$algebra and fuzzy algebra, where $\mathcal{X} \neq \emptyset$. A nonempty class $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ is called a fuzzy σ−algebra over a fuzzy set $\mathcal{X}^*$, if

1. $\emptyset^* \in \mathcal{H}^*$, where $\emptyset^* = \{(\omega, 0) : \forall \omega \in \mathcal{X}\}$.
2. If $E \in \mathcal{H}^*$, then $E^c \in \mathcal{H}^*$.
3. If $E_1, E_2, \ldots \in \mathcal{H}^*$, then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{H}^*$.

If condition 3 is satisfied only for finite sets, then $\mathcal{H}^*$ is said to be a fuzzy algebra over a fuzzy set $\mathcal{X}^*$.

In this work we introduce the concept of fuzzy σ− ring and fuzzy ring which are generalizations of fuzzy $\sigma-$algebra. The main goal of this paper is to study these concepts, give basic properties, examples, characterizations and studied relationships between them.

2. Preliminaries

This section is going to review some well-known definitions in measure theory.

Definition 2.1 [3]

Let $\mathcal{X} \neq \emptyset$. A collection $\mathcal{H}$ is called σ− ring iff :

1. If $F, E \in \mathcal{H}$, then $F \setminus E \in \mathcal{H}$.
2. If $E_1, E_2, \ldots \in \mathcal{H}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{H}$.

Definition 2.2 [1]

Let $\mathcal{X} \neq \emptyset$. A collection $\mathcal{H}$ is called σ− field iff :

1. $\mathcal{X} \in \mathcal{H}$.
2. If $F \in \mathcal{H}$, then $F^c \in \mathcal{H}$.
3. If $E_1, E_2, \ldots \in \mathcal{H}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{H}$.

Proposition 2.3 [5]

Every σ−field is a σ− ring.

Definition 2.4 [8 , 9]

Let $\mathcal{X}$ be a nonempty set. Then:

1. The collection of all fuzzy sets in $\mathcal{X}$ is called a fuzzy power set and is denoted by $\mathcal{P}^*(\mathcal{X})$, In symbols: $\mathcal{P}^*(\mathcal{X}) = \{F : F \text{ is a fuzzy set in } \mathcal{X}\}$.
2. The empty fuzzy set in $\mathcal{X}$ is denoted by $\emptyset^*$ and defined as: $\emptyset^* = \{(\omega, 0) : \forall \omega \in \mathcal{X}\}$.
3. The fuzzy set $\mathcal{X}^*$ in $\mathcal{X}$ is defined as: $\mathcal{X}^* = \{(\omega, 1) : \forall \omega \in \mathcal{X}\}$.

Definition 2.5 [7]

Let $\mathcal{X}$ be a nonempty set. Then the union of the two fuzzy sets F and E in $\mathcal{X}$ with respective membership functions $\upsilon_F(\omega)$ and $\upsilon_E(\omega)$ is a fuzzy set G in $\mathcal{X}$ whose membership function is related to those of F and E by $\upsilon_G(\omega) = \max_{\omega \in \mathcal{X}} \{\upsilon_F(\omega), \upsilon_E(\omega)\}$, In symbols:

$\mathrm{G} = \mathrm{F} \cup \mathrm{E} \Leftrightarrow \mathrm{G} = \{(\omega, \max_{\omega \in \mathcal{X}} \{\upsilon_{\mathrm{F}}(\omega), \upsilon_{\mathrm{E}}(\omega)\}) : \omega \in \mathcal{X}\}$.

Definition 2.6 [8]

Let $\mathcal{X}$ be a nonempty set. Then the intersection of two fuzzy sets F and E in $\mathcal{X}$ with respective membership functions $\upsilon_{\mathrm{F}}(\omega)$ and $\upsilon_{\mathrm{E}}(\omega)$ is a fuzzy set G in $\mathcal{X}$ whose membership function is related to those of F and E by $\upsilon_{\mathrm{G}}(\omega) = \min_{\omega \in \mathcal{X}} \{\upsilon_{\mathrm{F}}(\omega), \upsilon_{\mathrm{E}}(\omega)\}$, In symbols:

$\mathrm{G} = \mathrm{F} \cap \mathrm{E} \Leftrightarrow \mathrm{G} = \{(\omega, \min_{\omega \in \mathcal{X}} \{\upsilon_{\mathrm{F}}(\omega), \upsilon_{\mathrm{E}}(\omega)\}) : \omega \in \mathcal{X}\}$.

Definition 2.7 [9]

Let $\mathcal{X}$ be a nonempty set and F is a fuzzy set in $\mathcal{X}$. Then the complement of a fuzzy set F is denoted by F^{c} and defined as: $\mathrm{F}^{\mathrm{c}} = \{(\omega, 1 - \upsilon_{\mathrm{F}}(\omega)) : \omega \in \mathcal{X}\}$.

Proposition 2.8 [10]

Every fuzzy σ–algebra is a fuzzy algebra.

3. The main results:

In this section, the basic definitions and facts related to this work are recalled, which starts with the following definition**.**

Definition 3.1

Let $\mathcal{X} \neq \emptyset$. A collection $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ is a fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$, iff

4. $\emptyset^* \in \mathcal{H}^*$.
5. If $\mathrm{F}, \mathrm{E} \in \mathcal{H}^*$, then $\mathrm{F} \setminus \mathrm{E} \in \mathcal{H}^*$.
6. If $\mathrm{E}_1, \mathrm{E}_2, \ldots \in \mathcal{H}^*$, then $\bigcup_{k=1}^{\infty} \mathrm{E}_{\mathrm{k}} \in \mathcal{H}^*$.

Definition 3.2

A fuzzy measurable space relatively to a fuzzy σ– ring is an ordered pair ($\mathcal{X}^*$, $\mathcal{H}^*$), where $\mathcal{X} \neq \emptyset$and $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ be a fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$ and an element of $\mathcal{H}^*$ is called a measurable set relatively to fuzzy σ– ring.

Example 3.1

Let $\mathcal{X} \neq \emptyset$. Then each of $\emptyset^*$ and $\mathcal{P}^*(\mathcal{X})$ is a fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$.

Example 3.2

Assume $\mathcal{X} = \{a, b\}$ and $\mathcal{H}^* = \{\mathrm{E}_{\mathrm{k}}, \mathrm{E}_{\mathrm{k}}^{\mathrm{c}} \subset \mathcal{X}^* : \mathrm{E}_{\mathrm{k}} \supset \mathrm{E}_{\mathrm{k}+1}$, for every $\mathrm{k} = 1,2, \ldots\}$.
Then $\emptyset^*, \mathcal{X}^* \in \mathcal{H}^*$. If $\mathrm{F}, \mathrm{E} \in \mathcal{H}^*$, then $\mathrm{F}^{\mathrm{c}}, \mathrm{E}^{\mathrm{c}} \in \mathcal{H}^*$ and either $\mathrm{E} \subset \mathrm{F}^{\mathrm{c}}$ or $\mathrm{E} \supset \mathrm{F}^{\mathrm{c}}$. If $\mathrm{E} \subset \mathrm{F}^{\mathrm{c}}$, then $\mathrm{F} \subset \mathrm{E}^{\mathrm{c}}$, hence $\mathrm{F} \cap \mathrm{E}^{\mathrm{c}} \in \mathcal{H}^*$, that is $\mathrm{F} \setminus \mathrm{E} \in \mathcal{H}^*$.
If $\mathrm{E} \supset \mathrm{F}^{\mathrm{c}}$, then $\mathrm{E}^{\mathrm{c}} \subset \mathrm{F}$, hence $\mathrm{F} \cap \mathrm{E}^{\mathrm{c}} \in \mathcal{H}^*$, that is $\mathrm{F} \setminus \mathrm{E} \in \mathcal{H}^*$.
Now, If $\mathrm{E}_1, \mathrm{E}_2, \ldots \in \mathcal{H}^*$, then $\mathrm{E}_{\mathrm{k}} \supset \mathrm{E}_{\mathrm{k}+1}$ for every $(\mathrm{k} = 1,2, \ldots)$ and hence $\upsilon_{\mathrm{E}_{\mathrm{k}}}(\omega) > \upsilon_{\mathrm{E}_{\mathrm{k}+1}}(\omega)$ for all $\omega \in \mathcal{X}$ and hence

$$\bigcup_{k=1}^{\infty} \mathrm{E}_{\mathrm{k}} = \{(\omega, \sup\{\upsilon_{\mathrm{E}_1}(\omega), \upsilon_{\mathrm{E}_2}(\omega), \upsilon_{\mathrm{E}_3}(\omega), \ldots\} : \forall \omega \in \mathcal{X})\}$$
$$= \left\{\left(\omega, \upsilon_{\mathrm{E}_1}(\omega)\right) : \forall \omega \in \mathcal{X}\right\} = \mathrm{E}_1$$

Thus, $\bigcup_{k=1}^{\infty} \mathrm{E}_{\mathrm{k}} \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ is a fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$ and hence ($\mathcal{X}^*$, $\mathcal{H}^*$) is a fuzzy measurable space relatively to the fuzzy σ– ring.

Example 3.3

Let $\mathcal{X}$ ={ a ,b ,c} and $\mathcal{H}^*$ ={ $\emptyset^*$,{(a,0.3),(b,0.3),(c,0.4)},{(a,0.4),(b,0.4),(c,0.2)}}. Then $\mathcal{H}^*$ is not a fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$, because {(a,0.3),(b,0.3),(c,0.4)}, {(a,0.4),(b,0.4),(c,0.2)} ∈ $\mathcal{H}^*$, but
$\{(a, 0.3), (b, 0.3), (c, 0.4)\} \cup \{(a, 0.4), (b, 0.4), (c, 0.2)\}$ ={(a,0.4),(b,0.4),(c,0.4)}∉ $\mathcal{H}^*$.

Lemma 3.1

Let $\{\mathcal{H}_i^*\}_{i\in I}$ be a nonempty collection of fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$. Then $\bigcap_{i\in I}\mathcal{H}_i^*$ is a fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$.

Proof

Since $\mathcal{H}_i^*$ is a nonempty collection of a fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$ $\forall i \in I$, then there is $\emptyset^* \neq E \in \mathcal{H}_i^*$ $\forall i \in I$, which implies that $E \in \bigcap_{i\in I}\mathcal{H}_i^*$ and hence $\bigcap_{i\in I}\mathcal{H}_i^* \neq \emptyset^*$. It is clear that $\emptyset^* \in \bigcap_{i\in I}\mathcal{H}_i^*$. Let $F, E \in \bigcap_{i\in I}\mathcal{H}_i^*$. Then $F, E \in \mathcal{H}_i^*$ $\forall i \in I$, hence $F\backslash E \in \mathcal{H}_i^*$ $\forall i \in I$. Therefore $F\backslash E \in \bigcap_{i\in I}\mathcal{H}_i^*$. Let $E_1, E_2, \ldots \in \bigcap_{i\in I}\mathcal{H}_i^*$, then $E_1, E_2, \ldots \in \mathcal{H}_i^*$ $\forall i \in I$ and hence $\bigcup_{k=1}^{\infty} E_k \in \mathcal{H}_i^*$ $\forall i \in I$, thus $\bigcup_{k=1}^{\infty} E_k \in \bigcap_{i\in I}\mathcal{H}_i^*$. Therefore, $\bigcap_{i\in I}\mathcal{H}_i^*$ is a fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$.

In the following example shows the union for two fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$ needs not be a fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$.

Example 3.4

Let $\mathcal{X}$ ={ ω_1 , ω_2} and $\mathcal{H}_1^*$ ={$\emptyset^*$,{(ω_1,0.1),(ω_2,0.5)} , {(ω_1,0.9),(ω_2 ,0.5)} , $\mathcal{X}^*$}, $\mathcal{H}_2^*$ ={$\emptyset^*$,{(ω_1,0.2),(ω_2,0.4)} , {(ω_1,0.8),(ω_2,0.6)} , $\mathcal{X}^*$}. Then $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$ are fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$. Now,

$$\mathcal{H}_1^* \cup \mathcal{H}_2^* = \begin{Bmatrix} \emptyset^*, \{(\omega_1, 0.1), (\omega_2, 0.5)\}, \{(\omega_1, 0.9), (\omega_2, 0.5)\}, \\ \{(\omega_1, 0.2), (\omega_2, 0.4)\}, \{(\omega_1, 0.8), (\omega_2, 0.6)\}, \mathcal{X}^* \end{Bmatrix}.$$

Put, $E_1 = \{(\omega_1, 0.1), (\omega_2, 0.5)\}$, $E_2 = \{(\omega_1, 0.9), (\omega_2, 0.5)\}$, $E_3 = \{(\omega_1, 0.2), (\omega_2, 0.4)\}$, $E_4 = \{(\omega_1, 0.8), (\omega_2, 0.6)\}$ .Then $E_k \in \mathcal{H}_1^* \cup \mathcal{H}_2^*$ for all k=1,2,…,4. So, we have

$$\bigcup_{k=1}^{4} E_k = \begin{Bmatrix} (\omega_1, Max\{0.1,0.9,0.2,0.8\}), \\ (\omega_2, Max\{0.5,0.5,0.4,0.6\}), \\ \end{Bmatrix}$$

$$=\{(\omega_1,0.9),(\omega_2,0.6)\} \notin \mathcal{H}_1^* \cup \mathcal{H}_2^* .$$

Thus, $\mathcal{H}_1^* \cup \mathcal{H}_2^*$ is not fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$.

Definition 3.3

Assume $\mathcal{X} \neq \emptyset$ and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$, then the intersection of all fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$, which includes $\mathfrak{T}^*$ is said to be the fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$ generated by $\mathfrak{T}^*$ and denoted by $\sigma^r(\mathfrak{T}^*)$, that is:

$\sigma^r(\mathfrak{T}^*) = \bigcap\{\mathcal{H}_i^* : \mathcal{H}_i^*$ is a fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$and $\mathcal{H}_i^* \supseteq \mathfrak{T}^*, \forall i \in I\}$.

Proposition 3.1

Assume $\mathcal{X} \neq \emptyset$ and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $\sigma^r(\mathfrak{T}^*)$ is the smallest fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$.

Proof

The result is directed by the definition of $\sigma^r(\mathfrak{T}^*)$ and Lemma 3.1.

Example 3.5

Let $\mathcal{X}$ ={ ω_1 , ω_2} and $\mathfrak{T}^* = \{ \{(\omega_1, 0.1), (\omega_2, 0.5)\}, \{(\omega_1, 0.9), (\omega_2, 0.5)\}\}$. Then, $\sigma^r(\mathfrak{T}^*)$ ={$\emptyset^*$,{(ω_1,0.1),(ω_2,0.5)},{(ω_1,0.9),(ω_2,0.5)},$\mathcal{X}^*$} is the smallest fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$ that include $\mathfrak{T}^*$.

Proposition 3.2

Assume $\mathcal{X} \neq \emptyset$ and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$, then $\sigma^r(\mathfrak{T}^*) = \mathfrak{T}^*$ if and only if $\mathfrak{T}^*$ is fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$.

Proof

The result direct by definition of $\sigma^r(\mathfrak{T}^*)$ and Proposition 3.1.

Proposition 3.3

Every fuzzy σ– algebra over a fuzzy set $\mathcal{X}^*$ is a fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$.

Proof

Let $\mathcal{H}^*$ be a fuzzy σ– algebra over a fuzzy set $\mathcal{X}^*$. Then from the definition of fuzzy σ– algebra we get, $\emptyset^* \in \mathcal{H}^*$. Let $\mathrm{F}, \mathrm{E} \in \mathcal{H}^*$. Then $\mathrm{E}^c \in \mathcal{H}^*$, hence $\mathrm{F} \cap \mathrm{E}^c \in \mathcal{H}^*$, but $\mathrm{F} \cap \mathrm{E}^c = \mathrm{F} \backslash \mathrm{E}$ implies that $\mathrm{F} \backslash \mathrm{E} \in \mathcal{H}^*$.

Let $\mathrm{E}_1, \mathrm{E}_2, \ldots \in \mathcal{H}^*$. Then by definition of fuzzy σ– algebra we have, ${\mathrm{E}_\mathrm{k}}^c$ (for all $\mathrm{k} = 1,2, \ldots$) and $\bigcap_{\mathrm{k}=1}^{\infty} {\mathrm{E}_\mathrm{k}}^c \in \mathcal{H}^*$ and $(\bigcap_{\mathrm{k}=1}^{\infty} {\mathrm{E}_\mathrm{k}}^c)^c \in \mathcal{H}^*$. By De-Morgan law, we get $(\bigcap_{\mathrm{k}=1}^{\infty} {\mathrm{E}_\mathrm{k}}^c)^c = \bigcup_{\mathrm{k}=1}^{\infty} \mathrm{E}_\mathrm{k}$, hence $\bigcup_{\mathrm{k}=1}^{\infty} \mathrm{E}_\mathrm{k} \in \mathcal{H}^*$.

Therefore, $\mathcal{H}^*$ be a fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$.

While the converse is not true as shown below:

Example 3.6

Let $\mathcal{X}$ ={ ω_1 , ω_2} and $\mathcal{H}^* = \begin{Bmatrix} \emptyset^*, \{(\omega_1, 0), (\omega_2, 0.5)\}, \\ \{(\omega_1, 0.6), (\omega_2, 0.5)\}, \\ \{(\omega_1, 0.4), (\omega_2, 0.5)\} \end{Bmatrix}$. Put

F={(ω_1,0), (ω_2,0.5)}, then $\mathrm{F} \backslash \emptyset^* = \{(\omega_1, Min\{0,1-0\}), (\omega_2, Min\{0.5,1-0\})\}$

$= \{(\omega_1, 0), (\omega_2, 0.5)\}$= F

In the same way, we get $\emptyset^* \backslash \mathrm{F} = \emptyset^*$. Since $\mathrm{F^c} = \{(\omega_1, 1), (\omega_2, 0.5)\}$, then $\upsilon_F(\omega) \leq \upsilon_{\mathrm{F^c}}(\omega)$, hence $\mathrm{F} \subseteq \mathrm{F^c}$ and $\mathrm{F} \backslash \mathrm{F} = \mathrm{F}$. If E = {(ω_1,0.6),(ω_2,0.5)}, then $\mathrm{E} \backslash \emptyset^* = \mathrm{E}$ and $\emptyset^* \backslash \mathrm{E} = \emptyset^*$ and

If F = {(ω_1,0), (ω_2,0.5)} and E = {(ω_1,0.6),(ω_2,0.5)}, then $\mathrm{E^c}$ ={(ω_1,0.4),(ω_2,0.5)} and

$\mathrm{E} \backslash \mathrm{E} = \{(\omega_1, Min\{0.6,0.4\}), (\omega_2, Min\{0.5,0.5\})\}$

$= \{(\omega_1, 0.4), (\omega_2, 0.5)\} = \mathrm{E^c}$

$\mathrm{F} \backslash \mathrm{E} = \{(\omega_1, Min\{0,1-0.6\}), (\omega_2, Min\{0.5,1-0.5\})\}$

$= \{(\omega_1, 0), (\omega_2, 0.5)\}$= F

Similarly, $\mathrm{E} \backslash \mathrm{F} = \mathrm{E}$ and $\mathrm{E^c} \backslash \mathrm{F} = \mathrm{E^c}$. .

Now, $\mathrm{F} \cup \mathrm{E} \cup \mathrm{E^c} = \{(\omega_1, Sup\{0,0,0.6,0.4\}), (\omega_2, Sup\{0,0.5,0.5,0.5\})\}$

$= \{(\omega_1, 0.6), (\omega_2, 0.5)\}$= E.

Therefore, $\mathcal{H}^*$ is a fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$. In contrast, $\mathcal{H}^*$ is not fuzzy σ– algebra over a fuzzy set $\mathcal{X}^*$, because{(ω_1,0),(ω_2,0.5)}∈ $\mathcal{H}^*$, but

$\{(\omega_1, 0), (\omega_2, 0.5)\}^{\mathrm{c}} = \{(\omega_1, 1), (\omega_2, 0.5)\} \notin \mathcal{H}^*$.

Theorem 3.1

Assume $\mathcal{X} \neq \emptyset$ and $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ with $\mathcal{X}^* \in \mathcal{H}^*$. Then $\mathcal{H}^*$ is fuzzy σ− algebra over a fuzzy set $\mathcal{X}^*$ if and only if $\mathcal{H}^*$ is a fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$.

Proof

Assume that $\mathcal{H}^*$ is fuzzy σ− algebra over a fuzzy set $\mathcal{X}^*$, then by Proposition 3.3 we get $\mathcal{H}^*$ is fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$.

Conversely: Suppose that $\mathcal{H}^*$ is a fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$ such that $\mathcal{X}^* \in \mathcal{H}^*$, then $\emptyset^* \in \mathcal{H}^*$. Let $\mathrm{E} \in \mathcal{H}^*$. Then $\mathcal{X}^* \setminus \mathrm{E} \in \mathcal{H}^*$, but

$$\mathcal{X}^* \setminus \mathrm{E} = \mathcal{X}^* \cap \mathrm{E}^c = \left\{\left(\omega, \mathrm{Min}\{\upsilon_{\mathcal{X}^*}, \upsilon_{\mathrm{E}^c}(\omega)\}\right) : \forall \omega \in \mathcal{X}\right\}$$
$$= \left\{\left(\omega, \mathrm{Min}\{1, 1 - \upsilon_{\mathrm{E}}(\omega)\}\right) : \forall \omega \in \mathcal{X}\right\}$$
$$= \left\{\left(\omega, 1 - \upsilon_{\mathrm{E}}(\omega)\right) : \forall \omega \in \mathcal{X}\right\} = \mathrm{E}^c$$

This implies that, $\mathcal{X}^* \setminus \mathrm{E} = \mathrm{E}^c \in \mathcal{H}^*$. Let $\mathrm{E}_1, \mathrm{E}_2, \ldots \in \mathcal{H}^*$. Then as shown above we have, ${\mathrm{E}_k}^c$ (for all $k = 1,2,\ldots$). Hence by definition of a fuzzy σ− ring we get $\bigcup_{k=1}^{\infty} {\mathrm{E}_k}^c \in \mathcal{H}^*$, thus $(\bigcup_{k=1}^{\infty} {\mathrm{E}_k}^c)^c \in \mathcal{H}^*$. By De-Morgan law, we get $(\bigcup_{k=1}^{\infty} {\mathrm{E}_k}^c)^c = \bigcap_{k=1}^{\infty} \mathrm{E}_k$ hence $\bigcap_{k=1}^{\infty} \mathrm{E}_k \in \mathcal{H}^*$.

Therefore, $\mathcal{H}^*$ be a fuzzy σ− algebra over a fuzzy set $\mathcal{X}^*$.

Proposition 3.4

Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $\sigma^r(\mathfrak{T}^*) \subseteq \sigma(\mathfrak{T}^*)$, where $\sigma(\mathfrak{T}^*)$ is the smallest fuzzy σ–algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$.

Proof

Clearly.

Proposition 3.5

Let $\mathcal{X} \neq \emptyset$ and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. If $\mathcal{X}^* \in \sigma^r(\mathfrak{T}^*)$, then $\sigma^r(\mathfrak{T}^*) = \sigma(\mathfrak{T}^*)$.

Proof

The proof follows from Theorem 3.1.

Definition 3.4

Let assume $\mathcal{X} \neq \emptyset$. A class $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ is said to be a fuzzy ring over a fuzzy set $\mathcal{X}^*$, if

1. $\emptyset^* \in \mathcal{H}^*$.
2. If $\mathrm{F}, \mathrm{E} \in \mathcal{H}^*$, then $\mathrm{F} \setminus \mathrm{E} \in \mathcal{H}^*$.
3. If $\mathrm{E}_1, \mathrm{E}_2,, \mathrm{E}_n \in \mathcal{H}^*$, then $\bigcup_{k=1}^{n} \mathrm{E}_k \in \mathcal{H}^*$.

Definition 3.5

A fuzzy measurable space relatively to fuzzy ring is an ordered pair ($\mathcal{X}^*$, $\mathcal{H}^*$), where $\mathcal{X}$ is a nonempty set and $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ be a fuzzy ring over a fuzzy set $\mathcal{X}^*$ and an element of $\mathcal{H}^*$ is called a measurable set relatively to fuzzy ring.

Example 3.7

Suppose $\mathcal{X} \neq \emptyset$. Then each of $\emptyset^*$ and $\mathcal{P}^*(\mathcal{X})$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

Example 3.8

Let $\mathcal{X}$ ={ ω_1 , ω_2} and $\mathcal{H}^*$ ={$\emptyset^*$,{(ω_1,0.4),(ω_2,0.3)} , {(ω_1,0.6),(ω_2,0.7)}}. Then $\mathcal{H}^*$ and is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

Example 3.9

Let $\mathcal{X}$ ={ ω_1 , ω_2} and $\mathcal{H}^*$ ={$\emptyset^*$,{(ω_1,0.3),(ω_2,0.7)},{(ω_1,0.2),(ω_2,0.8)}, $\mathcal{X}^*$}. Then $\mathcal{H}^*$ is not a fuzzy ring over a fuzzy set $\mathcal{X}^*$, because

{(ω_1,0.3),(ω_2,0.7)} and {(ω_1,0.2),(ω_2,0.8)} $\in \mathcal{H}^*$, but

$\{(\omega_1, 0.3), (\omega_2, 0.7)\} \cup \{(\omega_1, 0.2), (\omega_2, 0.8)\} = \{(\omega_1, 0.3), (\omega_2, 0.8)\} \notin \mathcal{H}^*$.

Lemma 3.2

Let $\{\mathcal{H}_i^*\}_{i\in I}$ be a nonempty collection of a fuzzy ring over a fuzzy set $\mathcal{X}^*$. Then $\bigcap_{i\in I}\mathcal{H}_i^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

Proof

Direct.

Definition 3.6

Assume $\mathcal{X} \neq \emptyset$ and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then the intersection of all fuzzy rings over a fuzzy set $\mathcal{X}^*$, which includes $\mathfrak{T}^*$ is said to be the fuzzy ring over a fuzzy set $\mathcal{X}^*$ generated by $\mathfrak{T}^*$ and denoted by $R(\mathfrak{T}^*)$, that is,

$R(\mathfrak{T}^*)= \bigcap\{\mathcal{H}_i^*: \mathcal{H}_i^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$and $\mathcal{H}_i^* \supseteq \mathfrak{T}^*, \forall i \in I\}$.

Proposition 3.6

If $\mathcal{X} \neq \emptyset$ and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $R(\mathfrak{T}^*)$ is the smallest fuzzy ring over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$.

Proof

The result is directed by the definition of $R(\mathfrak{T}^*)$ and Lemma 3.2.

Proposition 3.7

Every fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

Proof

Let $\mathcal{H}^*$ be a fuzzy σ– ring over a fuzzy set $\mathcal{X}^*$. Then by definition of fuzzy σ– ring we have, $\emptyset^* \in \mathcal{H}^*$. Let $F, E \in \mathcal{H}^*$. Then $F \backslash E \in \mathcal{H}^*$.

Let $E_1, E_2, E_n \in \mathcal{H}^*$. Consider, $E_m = \emptyset^*$ for all $m > n$, then we get $E_1, E_2, E_3, \ldots \in \mathcal{H}^*$ and hence from the definition of fuzzy σ– ring, we have

$\bigcup_{k=1}^{\infty} E_k \in \mathcal{H}^*$, but

$\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{n} E_k \cup E_{n+1} \cup E_{n+2} \cup \ldots = \bigcup_{k=1}^{n} E_k \cup \emptyset^* \cup \emptyset^* \cup \ldots = \bigcup_{k=1}^{n} E_k$. Thus $E_1 \cup E_2 \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

In general, the converse of the above proposition is not true as shown in the following example:

Example 3.10

Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{J}$ = finite disjoint union of right – semi-closed intervals. Assume that $\mathcal{H}^* = \{$ all ($\mathcal{J}, \upsilon_{\mathcal{J}}$) }. Then $\mathcal{H}^*$ is a fuzzy ring over a fuzzy set $\mathbb{R}^*$, but $\mathcal{H}^*$ is not fuzzy σ– ring over a fuzzy set $\mathbb{R}^*$. Because if we take E_k = {((0, 1- (1/k)], $\mathcal{v}_{E_k}$) }, k=1,2,…, then $E_k \in \mathcal{H}^*$ ∀n, but $\bigcup_{k=1}^{\infty} E_k$ = {((0,1), $\mathcal{v}_{E_k}$)} $\notin \mathcal{H}^*$.

Proposition 3.8

Assume $\mathcal{X} \neq \emptyset$ and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $R(\mathfrak{T}^*) \subseteq \sigma^r(\mathfrak{T}^*)$.

Proof

The proof is directed by proposition 3.7 and proposition 3.6.

Proposition 3.9

Every fuzzy algebra over a fuzzy set $\mathcal{X}^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

Proof

Let $\mathcal{H}^*$ be a fuzzy algebra over a fuzzy set $\mathcal{X}^*$. Then by the definition of fuzzy algebra we have, $\mathcal{X}^* \in \mathcal{H}^*$, hence $\emptyset^* = \mathcal{X}^{*c} \in \mathcal{H}^*$ Let $F, E \in \mathcal{H}^*$. Then $E^c \in \mathcal{H}^*$, hence $F \cap E^c \in \mathcal{H}^*$, but $F \cap E^c = F \setminus E$ implies that $F \setminus E \in \mathcal{H}^*$. Let $E_1, E_2 \in \mathcal{H}^*$. Then by definition of fuzzy algebra implies that $E_1 \cup E_2 \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ be a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

while the converse is not true as shown in the next example:

Example 3.11

Let $\mathcal{X}$ ={ a , b} and $\mathcal{H}^*$ ={$\emptyset^*$,{(a,0), (b,0.5)},{(a,0.6),(b,0.5)},{(a,0.4),(b,0.5)}}. Then, $\mathcal{H}^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$. In contrast, $\mathcal{H}^*$ is a fuzzy algebra over a fuzzy set $\mathcal{X}^*$, because {(a,0), (b,0.5)}∈ $\mathcal{H}^*$, but $\{(a, 0), (b, 0.5)\}^c = \{(a, 1), (b, 0.5)\} \notin \mathcal{H}^*$.

Proposition 3.10

Assume $\mathcal{X} \neq \emptyset$ and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$, then $R(\mathfrak{T}^*) \subseteq AL(\mathfrak{T}^*)$ where $AL(\mathfrak{T}^*)$ is the smallest fuzzy algebra over a fuzzy set $\mathcal{X}^*$ that include $\mathfrak{T}^*$.

Proof

The proof is followed by proposition 3.9 with proposition 3.6.

Proposition 3.11

Assume $\mathcal{X}$ to be a non-empty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $R(\mathfrak{T}^*) \subseteq \sigma(\mathfrak{T}^*)$.

Proof

Observe.

Theorem 3.2

Assume $\mathcal{X} \neq \emptyset$ and $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ such that $\mathcal{X}^* \in \mathcal{H}^*$. Then $\mathcal{H}^*$ is a fuzzy algebra over a fuzzy set $\mathcal{X}^*$ if and only if $\mathcal{H}^*$ is fuzzy ring over a fuzzy set $\mathcal{X}^*$.

Proof

Assume that $\mathcal{H}^*$ is fuzzy algebra over a fuzzy set $\mathcal{X}^*$, then by Proposition 3.9, we get $\mathcal{H}^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

Conversely: Suppose that $\mathcal{H}^*$ is fuzzy ring over a fuzzy set $\mathcal{X}^*$ such that $\mathcal{X}^* \in \mathcal{H}^*$. Let $\mathrm{E} \in \mathcal{H}^*$. Then $\mathcal{X}^* \backslash \mathrm{E} \in \mathcal{H}^*$, but

$$\mathcal{X}^* \backslash \mathrm{E} = \mathcal{X}^* \cap \mathrm{E}^c = \{(\omega, \mathrm{Min}\{\mathcal{v}_{\mathcal{X}^*}, \mathcal{v}_{\mathrm{E}^c}(\omega)\}) : \forall \omega \in \mathcal{X}\}$$
$$= \{(\omega, \mathrm{Min}\{1, 1 - \mathcal{v}_{\mathrm{E}}(\omega)\}) : \forall \omega \in \mathcal{X}\}$$
$$= \{(\omega, 1 - \mathcal{v}_{\mathrm{E}}(\omega)) : \forall \omega \in \mathcal{X}\} = \mathrm{E}^c$$

Which implies that, $\mathcal{X}^* \backslash \mathrm{E} = \mathrm{E}^c \in \mathcal{H}^*$. Let $\mathrm{E}_1, \mathrm{E}_2, \dots, \mathrm{E}_n \in \mathcal{H}^*$. Then $\bigcup_{k=1}^{n} \mathrm{E}_k \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ be a fuzzy algebra over a fuzzy set $\mathcal{X}^*$.

Proposition 3.12

Suppose $\mathcal{X}$ be a non-empty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. If $\mathcal{X}^* \in \mathrm{R}(\mathfrak{T}^*)$, then $\mathrm{R}(\mathfrak{T}^*) = \mathrm{AL}(\mathfrak{T}^*)$.

Proof

The proof is followed by Theorem 3.2.

Proposition 3.13

Every fuzzy σ− algebra over a fuzzy set $\mathcal{X}^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

4. Conclusions

We will try to generalize the concept of fuzzy σ− ring to some other concepts in future works. We define the concept of measure on fuzzy σ− ring and discuss many properties of this concept. In this study, the concepts of fuzzy σ− ring and fuzzy ring over a fuzzy set $\mathcal{X}^*$ weakly are introduced as a generalization of fuzzy σ− algebra and fuzzy algebra over the same fuzzy set $\mathcal{X}^*$. Furthermore, some properties of these concepts are investigated such as:

1. every fuzzy σ− algebra over a fuzzy set $\mathcal{X}^*$ is a fuzzy σ− ring over fuzzy set $\mathcal{X}^*$.
2. Assume $\mathcal{X} \neq \emptyset$ and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $\sigma^r(\mathfrak{T}^*)$ is the smallest fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$ that include $\mathfrak{T}^*$.
3. Assume $\mathcal{X} \neq \emptyset$ and $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ such that $\mathcal{X}^* \in \mathcal{H}^*$. Then $\mathcal{H}^*$ is a fuzzy σ− algebra over a fuzzy set $\mathcal{X}^*$ if and only if $\mathcal{H}^*$ is fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$.
4. Every fuzzy σ− ring over a fuzzy set $\mathcal{X}^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.
5. Every fuzzy algebra over a fuzzy set $\mathcal{X}^*$ is a fuzzy ring over a fuzzy set $\mathcal{X}^*$.

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