IHJPAS. 36 (3) 2023

Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq

Best Proximity Point Theorem for $\widetilde{\varphi}-\widetilde{\psi}$ −Proximal Contractive Mapping in Fuzzy Normed Space

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Article history: Received 19 October 2022, Accepted 12 December 2022, Published in July 2023.

doi.org/10.30526/36.3.3080

Abstract

The study of fixed points on the maps fulfilling certain contraction requirements has several applications and has been the focus of numerous research endeavors. On the other hand, as an extension of the idea of the best approximation, the best proximity point (BPP) emerges. The best approximation theorem ensures the existence of an approximate solution; the best proximity point theorem is considered for addressing the problem in order to arrive at an optimum approximate solution. This paper introduces a new kind of proximal contraction mapping and establishes the best proximity point theorem for such mapping in fuzzy normed space ($\tilde{F}_N$ space). In the beginning, the concept of the best proximity point was introduced. The concept of proximal contractive mapping in the context of fuzzy normed space is then presented. Following that, the best proximity point theory for this kind of mapping is established. In addition, we provide an example application of the results.

Keywords: Fuzzy normed space, $\tilde{\varphi}-\tilde{\psi}$ −proximal contractive mapping, Best proximity point, Best proximity point theorems, Proximal contractive mapping .

1. Introduction and Preliminaries

According to the renowned Banach's contraction principle[1], any contraction self-mapping in a complete metric space has a unique fixed point. This idea has been expanded and generalized in numerous approaches. Consider $\Lambda: \widetilde{L} \rightarrow \widetilde{D}$ to be a non-self mapping, $\widetilde{L}$ and $\widetilde{D}$ are subsets of a metric space (Q, d). There may or may not be a solution to the equation $\Lambda\omega = \omega$. The presence of an approximate solution that is optimum is examined via the best proximity point (BPP) theorems. An optimum approximation solution ω is one for which the error d(ω; $\Lambda\omega$) is small, and this is the goal of BPP theorem. These optimal approximation solutions are referred to as the best proximity points (BPP) of Λ.

On the other hand, Katsaras [2] was a pioneer in establishing fuzzy norms in linear spaces. Numerous articles on fuzzy normed spaces have been published; see [3–9].

In this paper, the notion of $\tilde{\varphi}-\tilde{\psi}-$ proximal contractive mapping (briefly, $\tilde{\varphi}-\tilde{\psi}-$ PC mapping) in a fuzzy normed space is presented, as well as the proof of the best proximity point theorem for this mapping. An illustration, in the form of examples, is offered to demonstrate how significant the results are.

For completeness, we provide some fundamental notions.

Definition 1.1[10]: Let Q be a vector space over a field R. A fuzzy normed space (briefly, $\tilde{F}_N$ space) refers to the triplet $(Q, \tilde{F}_N, \circledast)$ where $\circledast$ represents a t-norm, $\tilde{F}_N$ is a fuzzy set on $Q \times R$ fulfilling the requirements below for each $\omega, \rho \in Q$.

$(\tilde{F}_{N\,1})\tilde{F}_N(\omega, 0) = 0,$

$(\tilde{F}_{N\,2})\tilde{F}_N(\omega, \tau) = 1, \forall \tau > 0$ if and only if $\omega = 0,$

$(\tilde{F}_{N\,3})\tilde{F}_N(\gamma\omega, \tau) = \tilde{F}_N(\omega, \tau/|\gamma|), \forall\, 0 \neq \gamma \in R, \tau \geq 0$

$(\tilde{F}_{N\,4})$ $\tilde{F}_N(\omega, \tau) \circledast \tilde{F}_N(\omega, s) \leq \tilde{F}_N(\omega + \rho, \tau + s), \forall \tau, s \geq 0$

$(\tilde{F}_{N\,5})$ $\tilde{F}_N(\omega, .)$ $is\ left\ continuous$ for each $\omega \in Q,$ and $\tilde{F}_N(\omega, \tau) = 1$.

Definition 1.2[11]: Let $(Q, \tilde{F}_N, \circledast)$ be a $\tilde{F}_N$ space. A sequence $\{\omega_n\}$ is called convergent if $\exists \omega \in Q$, such that $\tilde{F}_N(\omega_n - \omega, \tau) = 1$ for each > 0 . And $\{\omega_n\}$ is called Cauchy sequence if $\tilde{F}_N\left(\omega_{n+j} - \omega_n, \tau\right) = 1$ for each $\tau > 0$ and $j = 1,2, ...$

Definition1.3[11]: If $(Q, \tilde{F}_N, \circledast)$ is a $\tilde{F}_N$ space. Then, if each Cauchy sequence in Q is convergent in Q, the $\tilde{F}_N$ space $(Q, \tilde{F}_N, \circledast)$ is termed to be complete.

In a fuzzy metric space $(Q, F_M, \circledast)$, C. Vetro and P. Salimi [12] presented the notion of fuzzy distance. Consider $\widetilde{L}\ and\ \widetilde{D}$ be nonempty subsets of $(Q, F_M, \circledast)$ and $\widetilde{L}_\circ(\tau)$, $\widetilde{D}_\circ(\tau)$ denoted by the sets:

$$\widetilde{L}_\circ(\tau) = \{\omega \in \widetilde{L} : F_M(\omega, \rho, \tau) = F_M(\widetilde{L}, \widetilde{D}, \tau) \text{ for some } \rho \in \widetilde{D}\}$$

$$\widetilde{D}_\circ(\tau) = \{\rho \in \widetilde{D} : F_M(\omega, \rho, \tau) = F_M(\widetilde{L}, \widetilde{D}, \tau) \text{ for some } \omega \in \widetilde{L}\}$$

Where, $F_M\left(\widetilde{L}, \widetilde{D}, \tau\right) = \sup\{F_M(\omega, \rho, \tau) : \omega \in \widetilde{L}, \rho \in \widetilde{D}\},$

The aforementioned notion is proposed in this paper in a $\tilde{F}_N$ space as follows:

Suppose that $\widetilde{L}$ and $\widetilde{D}$ are nonempty subsets of $(Q, \tilde{F}_N, \circledast)$ and $\widetilde{L}_\circ(\tau)$, $\widetilde{D}_\circ(\tau)$ are denoted by the following sets:

$\widetilde{L}_\circ(\tau) = \{\omega \in \widetilde{L} : \tilde{F}_N(\omega - \rho, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)$, for some $\rho \in \widetilde{D}\}$;

$\widetilde{D}_\circ(\tau) = \{\rho \in \widetilde{D} : \tilde{F}_N(\omega - \rho, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)$, for some $\omega \in \widetilde{L}\}$;

Where, $N_d(\widetilde{L}, \widetilde{D}, \tau) = sup\{\tilde{F}_N(\omega - \rho, \tau): \omega \in \widetilde{L}, \rho \in \widetilde{D}\}$.

2. Main Results

In this section, a new class of proximal contraction mapping, known as $\tilde{\varphi}-\tilde{\psi} - PC$ mapping is presented, followed by the proof of the BPP theorem for such a mapping in fuzzy normed space.

Definition 2.1: Let $(Q, \tilde{F}_N, \circledast)$ be a fuzzy Banach space and $\widetilde{L}$ and $\widetilde{D}$ be nonempty closed subsets of $(Q, \tilde{F}_N, \circledast)$. An element $\omega^\star \in \widetilde{L}$ is termed as ƁPP of a mapping $\Lambda: \widetilde{L} \to \widetilde{D}$ if it meets the criteria $\tilde{F}_N(\omega^\star - \Lambda\omega^\star, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)$ for each $\tau > 0$.

Let $\widetilde{\Psi}$ represent the set of all functions $\tilde{\psi}: [0, \infty) \to [0, 1]$ for which $\tilde{\psi}$ is a lower semicontinuous function and $\tilde{\psi}(\delta) = 1$ if and only if $\delta = 1$.

Let $\widetilde{\Phi}$ represent the set of all functions $\tilde{\varphi}: [0, \infty) \to [0, 1]$ where $\tilde{\varphi}$ is continuous and non-decreasing and $\tilde{\varphi}(\delta) = 1$ if and only if $\delta = 1$.

Definition 2.2: Let $(Q, \tilde{F}_N, \circledast)$ be a fuzzy normed space with two nonempty subsets $\widetilde{L}$ and $\widetilde{D}$. Then $\widetilde{D}$ is said to be approximatively compact with respect to $\widetilde{L}$ if each sequence $\{\rho_n\}$ in $\widetilde{D}$, meeting the condition $\tilde{F}_N(\omega - \rho_n, \tau) \to N_e(\omega, \widetilde{D}, \tau)$ where $N_e(\omega, \widetilde{D}, \tau) = {}^{sup}_{b \in \widetilde{D}} \{\tilde{F}_N(\omega - b, \tau)\}$ for all $\tau > 0$ and some ω in $\widetilde{L}$, has a convergent subsequence.

Definition 2.3: Let $\widetilde{L}$ and $\widetilde{D}$ be nonempty subsets of a fuzzy normed space $(Q, \tilde{F}_N, \circledast)$ and let $\Lambda: \widetilde{L} \to \widetilde{D}$ be a given mapping. A mapping Λ is called $\tilde{\varphi}-\tilde{\psi} - PC$ mapping if

$$\tilde{F}_N(u - \Lambda\omega, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)\ \tilde{F}_N(v - \Lambda\rho, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)\} \Rightarrow \tilde{\varphi}\left(\tilde{F}_N(u - v, \tau)\right) \geq \tilde{\varphi}\left(\tilde{F}_N(\omega - \rho, \tau)\right) + \tilde{\psi}\left(\tilde{F}_N(\omega - y, \tau)\right) \quad (1)$$

holds for each $\omega, \rho, u, v \in \widetilde{L}$, $\tau > 0$, $\tilde{\varphi} \in \widetilde{\Phi}$ and $\tilde{\psi} \in \widetilde{\Psi}$.

Below, we provide and illustrate our study results.

Theorem 2.4: Let $\widetilde{L}$ and $\widetilde{D}$ be nonempty subsets of a fuzzy Banach space $(Q, \tilde{F}_N, \circledast)$, $\widetilde{L}_\circ$ is nonempty and $\widetilde{D}$ is approximatively compact with respect to $\widetilde{L}$. Let $\Lambda: \widetilde{L} \to \widetilde{D}$ be $\tilde{\varphi}-\tilde{\psi} - PC$ mapping such that $\Lambda(\widetilde{L}_\circ(\tau)) \subseteq \widetilde{D}_\circ(\tau)$ for all $\tau > 0$.Then Λ possesses a unique ƁPP, that is, a unique $\omega^\star \in \widetilde{L}$ exists such that $\tilde{F}_N(\omega^\star - \Lambda\omega^\star, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)$.

Proof: Since $\widetilde{L}_\circ$ is not empty, we choose $\omega_\circ$ in $\widetilde{L}_\circ$. Taking $\Lambda\omega_\circ \in \Lambda(\widetilde{L}_\circ(\tau)) \subseteq \widetilde{D}_\circ(\tau)$ into account, we can find ω_1 in $\widetilde{L}_\circ$ such that $\tilde{F}_N(\omega_1 - \Lambda\omega_\circ, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)$. Further, since $\Lambda\omega_1 \in$

$\Lambda(\widetilde{L}_\circ(\tau)) \subseteq \widetilde{D}_\circ(\tau)$, consequently, an element ω_2 exists in $\widetilde{L}_\circ$ such that $\tilde{F}_N(\omega_2 - \Lambda\omega_1, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)$. By repeating the process, we get a sequence $\{\omega_n\}$ in $\widetilde{L}_\circ$ fulfills

$$\tilde{F}_N(\omega_{n+1} - \Lambda\omega_n, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau); \quad \text{for each } n \in N \cup \{0\}. \qquad (2)$$

Now employing (2) and using (1) with $u = \rho = \omega_n$, $v = \omega_{n+1}$ and $\omega = \omega_{n-1}$ yields the following result:

$$\tilde{\varphi}\left(\tilde{F}_N(\omega_n - \omega_{n+1}, \tau)\right) \geq \tilde{\varphi}\left(\tilde{F}_N(\omega_{n-1} - \omega_n, \tau)\right) + \tilde{\psi}\left(\tilde{F}_N(\omega_{n-1} - \omega_n, \tau)\right) \quad (3)$$

which implies $\tilde{F}_N(\omega_n - \omega_{n+1}, \tau) \geq \tilde{F}_N(\omega_{n-1} - \omega_n, \tau)$ for some $n \in N$ and hence $\{\tilde{F}_N(\omega_n - \omega_{n+1}, \tau)\}$ is an increasing sequence. Hence, $l(\tau) \in (0,1]$ exists such that $\tilde{F}_N(\omega_n - \omega_{n+1}, \tau) = l(\tau)$ for every $\tau > 0$. We will verify that $l(\tau) = 1$ for every $\tau > 0$. Assume that $0 < l(\tau_\circ) < 1$ where $\tau_\circ > 0$; then limiting as $n \to \infty$ in (3) yields,

$$\tilde{\varphi}\big(l(\tau_\circ)\big) \geq \tilde{\varphi}\big(l(\tau_\circ)\big) + \tilde{\psi}\big(l(\tau_\circ)\big)$$

and as a result, $\tilde{\psi}\big(l(\tau_\circ)\big) \leq 0$, which is contradictory. This means that for every $\tau > 0$, $l(\tau) = 1$.

Hence, $\tilde{F}_N(\omega_n - \omega_{n+1}, \tau) = 1 \qquad (4)$

Next, we demonstrate that $\{\omega_n\}$ is a Cauchy sequence. Contrarily, consider $\{\omega_n\}$ is not Cauchy sequence. Then $z \in (0,1)$ and $\tau_\circ > 0$ exists such that, for each $\kappa \geq 1$, $m(\kappa), n(\kappa) \in N$ exists with $m(\kappa) > n(\kappa) \geq \kappa$ and

$\leq 1 - z$

Consider $m(\kappa)$ is the least integer surpassing $n(k)$ meeting the preceding inequality, which means $that > 1 - z$

So, for every κ,

$$\begin{aligned} 1 - z &\geq \tilde{F}_N\left(\omega_{m(\kappa)} - \omega_{n(\kappa)}, \tau_\circ\right) \\ &\geq \tilde{F}_N\left(\omega_{m(\kappa)} - \omega_{m(\kappa)-1}, \tau_\circ\right) \circledast \tilde{F}_N\left(\omega_{m(\kappa)-1} - \omega_{n(\kappa)}, \tau_\circ\right) \\ &> \tilde{F}_N\left(\omega_{m(\kappa)} - \omega_{m(\kappa)-1}, \tau_\circ\right) \circledast 1 - z \end{aligned}$$

In the preceding inequality, if we take the limit to be $\kappa \to \infty$ and use (4), we acquire the following:

$$\tilde{F}_N\left(\omega_{m(\kappa)} - \omega_{n(\kappa)}, \tau_\circ\right) = 1 - z \qquad (5)$$

Now from

$$\tilde{F}_N\left(\omega_{m(\kappa)+1} - \omega_{n(\kappa)+1}, \tau_\circ\right) \geq \tilde{F}_N\left(\omega_{m(\kappa)+1} - \omega_{m(\kappa)}, \tau_\circ\right) \circledast \tilde{F}_N\left(\omega_{m(\kappa)} - \omega_{n(\kappa)}, \tau_\circ\right) \circledast \tilde{F}_N\left(\omega_{n(\kappa)} - \omega_{n(\kappa)+1}, \tau_\circ\right)$$

And,

$$\tilde{F}_N\left(\omega_{m(\kappa)} - \omega_{n(\kappa)}, \tau_\circ\right) \geq \tilde{F}_N\left(\omega_{m(\kappa)} - \omega_{m(\kappa)+1}, \tau_\circ\right) \circledast \tilde{F}_N\left(\omega_{m(\kappa)+1} - \omega_{n(\kappa)+1}, \tau_\circ\right) \circledast \tilde{F}_N\left(\omega_{n(\kappa)+1} - \omega_{n(\kappa)}, \tau_\circ\right)$$

it follows that

$$\tilde{F}_N\left(\omega_{m(\kappa)+1} - \omega_{n(\kappa)+1}, \tau_\circ\right) = 1 - z \qquad (6)$$

From (2), we know that

$$\{\tilde{F}_N\left(\omega_{m(\kappa)+1} - \Lambda\omega_{m(\kappa)}, \tau_\circ\right) = N_d(\widetilde{L}, \widetilde{D}, \tau_\circ)\ \tilde{F}_N\left(\omega_{n(\kappa)+1} - \Lambda\omega_{n(\kappa)}, \tau_\circ\right) = N_d(\widetilde{L}, \widetilde{D}, \tau_\circ)$$
(7)

Therfore , by (1) and (7), we have

$$\tilde{\varphi}\left(\tilde{F}_N\left(\omega_{m(\kappa)+1} - \omega_{n(\kappa)+1}, \tau_\circ\right)\right) \geq \tilde{\varphi}\left(\tilde{F}_N\left(\omega_{m(\kappa)} - \omega_{n(\kappa)}, \tau_\circ\right)\right) + \tilde{\psi}\left(\tilde{F}_N\left(\omega_{m(\kappa)} - \omega_{n(\kappa)}, \tau_\circ\right)\right)$$

Using $\kappa \to \infty$ as the limit in the previous inequality, we acquire:

$$\tilde{\varphi}(1 - z) \geq \tilde{\varphi}(1 - z) + \tilde{\psi}(1 - z)$$

which is a contradiction. Now, if $\tilde{\psi}(1 - z) = 1$, $\tilde{\psi}$'s property implies that z =0, which is a contradiction. Hence, $\{\omega_n\}$ is a Cauchy sequence. The sequence $\{\omega_n\}$ converges to some $\omega^* \in Q$ because of the completeness of $(Q, \tilde{F}_N, \circledast)$, which is,

$\tilde{F}_N(\omega_n - \omega^\star, \tau)$ = 1 for each $\tau > 0$.

On the other hand, we can write:

$$N_d(\widetilde{L}, \widetilde{D}, \tau) = \tilde{F}_N(\omega_{n+1} - \Lambda\omega_n, \tau)$$
$$\geq \tilde{F}_N(\omega_{n+1} - \omega^\star, \tau) \circledast \tilde{F}_N(\omega^\star - \Lambda\omega_n, \tau)$$
$$\geq \tilde{F}_N(\omega_{n+1} - \omega^\star, \tau) \circledast \tilde{F}_N(\omega^\star - \omega_{n+1}, \tau) \circledast \tilde{F}_N(\omega_{n+1} - \Lambda\omega_n, \tau)$$
$$= \tilde{F}_N(\omega_{n+1} - \omega^\star, \tau) \circledast \tilde{F}_N(\omega^\star - \omega_{n+1}, \tau) \circledast N_d(\widetilde{L}, \widetilde{D}, \tau)$$

which indicates

$$N_d(\widetilde{L},\widetilde{D},\tau) \geq \tilde{F}_N\,(\omega_{n+1} - \omega^\star,\tau) \circledast \tilde{F}_N\,(\omega^\star - \Lambda\omega_n,\tau)$$

$$\geq \tilde{F}_N\,(\omega_{n+1} - \omega^\star,\tau) \circledast \tilde{F}_N\,(\omega^\star - \omega_{n+1},\tau) \circledast N_d(\widetilde{L},\widetilde{D},\tau)$$

Using $n \to \infty$ as the limit in the previous inequality, we acquire:

$$N_d(\widetilde{L},\widetilde{D},\tau) \geq 1 \circledast \tilde{F}_N\,(\omega^\star - \Lambda\omega_n,\tau)$$

$$\geq 1 \circledast 1 \circledast N_d(\widetilde{L},\widetilde{D},\tau)$$

meaning,

$$\tilde{F}_N\,(\omega^\star - \Lambda x_n,\tau) = N_d(\widetilde{L},\widetilde{D},\tau)$$

Since $\widetilde{D}$ is approximatively compact with respect to $\widetilde{L}$, the sequence $\{\Lambda\omega_n\}$ has a subsequence $\{\Lambda\omega_{n_k}\}$ that converges to some $b \in \widetilde{D}$. Therefore,

$$\tilde{F}_N\,(\omega^\star - b,\tau) = \tilde{F}_N\left(\omega_{n_k+1} - \Lambda\omega_{n_k},\tau\right) = N_d(\widetilde{L},\widetilde{D},\tau)$$

And, so $\omega^\star \in \widetilde{L}_\circ(\tau)$. Since $\Lambda\omega^\star \in \Lambda(\widetilde{L}_\circ(\tau)) \subseteq \widetilde{D}_\circ(\tau)$, there exists $z \in \widetilde{L}_\circ(\tau)$ such that $\tilde{F}_N\,(z - \Lambda\omega^\star,\tau) = N_d(\widetilde{L},\widetilde{D},\tau)$. Accordingly, inequality (1) with $u = \omega_{n+1}$, $v = z$, $\omega = \omega_n$ *and* $\rho = \omega^\star$ demonstrates that

$$\tilde{\varphi}(\,\tilde{F}_N\,(\omega_{n+1} - z,\tau)) \geq \tilde{\varphi}\left(\tilde{F}_N\,(\omega_n - \omega^\star,\tau)\right) + \tilde{\psi}(\,\tilde{F}_N\,(\omega_n - \omega^\star,\tau)).$$

Letting $n \to \infty$ then we get

$$\tilde{\varphi}(\,\tilde{F}_N\,(\omega^\star - z,\tau)) \geq \tilde{\varphi}(1) + \tilde{\psi}(\,1) \geq 1.$$

This implies $\tilde{F}_N\,(\omega^\star - z,\tau) = 1$ for each $\tau > 0$, that is, $\omega^\star = z$ *and* $\tilde{F}_N\,(\omega^\star - \Lambda\omega^\star,\tau) = N_d(\widetilde{L},\widetilde{D},\tau)$. Now to prove uniqueness, suppose that $\zeta^* \neq \omega^\star$, such that $\tilde{F}_N\,(\omega^\star - \Lambda\omega^\star,\tau) = N_d\left(\widetilde{L},\widetilde{D},\tau\right)$ and $\tilde{F}_N\,(\zeta^* - \Lambda\zeta^*,\tau) = N_d\left(\widetilde{L},\widetilde{D},\tau\right)$. Now, using (1) with $u = \omega = \omega^\star$ and $v = \rho = \zeta^*$ we acquire,

$$\tilde{\phi}(\,\tilde{F}_N\,(\omega^\star - \zeta^*,\tau)) \geq \tilde{\varphi}\left(\tilde{F}_N\,(\omega^\star - \zeta^*,\tau)\right) + \tilde{\psi}(\tilde{F}_N\,(\omega^\star - \zeta^*,\tau)).$$

and so $\tilde{\psi}\left(\,\tilde{F}_N\,(\omega^\star - \zeta^*,\tau)\right) \leq 0$ which is a contradiction. Thus $\tilde{F}_N\,(\omega^\star - \zeta^*,\tau) = 1$ for all $\tau > 0$ and so $\omega^\star = \zeta^*$.

Example2.5: Consider $Q = R$ and $\tilde{F}_N : Q \times R \to [0,1]$ be a fuzzy norm specified by: $\tilde{F}_N\,(\,\omega,\tau) = e^{(-\frac{\|\omega\|}{\tau})}$, where $\omega \in Q$ and $\tau > 0$, and $\|\omega\|: R \to [0,\infty)$ such that $\|\omega\| = |\omega|$.

Let $\widetilde{L} = \{\,5,6,7\}$ *and* $\widetilde{D} = \{\,2,3,4\}$. So that,

$N_d(\widetilde{L},\widetilde{D},\tau) = sup\{\tilde{F}_N\,(\omega - \rho,\tau) : \omega \in \widetilde{L}, \rho \in \widetilde{D}\,\} = e^{(-\frac{1}{\tau})}$. Then $\widetilde{L}_\circ(\tau) = 5$ and $\widetilde{D}_\circ(\tau) = 4$.

Also, define $\Lambda: \widetilde{L} \rightarrow \widetilde{D}$ by

$\Lambda(\omega) = \{4, \quad if\ \omega = 5\ \omega - 2, \quad otherwise$

Observe that $\Lambda(\widetilde{L}\circ(\tau)) \subseteq \widetilde{D}\circ(\tau)$.

Also, assume that

$\{\ \tilde{F}_N(u - \Lambda\omega, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)\ \tilde{F}_N(v - \Lambda\rho, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)$

then we have

$u = v = 3.$

Now, we define $\tilde{\varphi}, \tilde{\psi} : [0, \infty) \rightarrow [0,1]$ by $\tilde{\varphi}(\delta) = 1 - \delta$ and $\tilde{\psi}(\delta) = \delta^2$ for each $\delta \in [0, 1]$ then we have

$$\tilde{F}_N(u - v, \tau) = e^{\left(-\frac{\|u-v\|}{\tau}\right)}$$

$$= e^{\left(-\frac{|3-3|}{\tau}\right)} = 1 \geq \tilde{\varphi}\left(\tilde{F}_N(\omega - \rho, \tau)\right) + \tilde{\psi}(\tilde{F}_N(\omega - \rho, \tau))$$

for all $\tau > 0$.

As a result, all of the requirements of Theorem 2.4 are met and the mapping Λ possesses a unique ƁPP. The unique ƁPP of Λ in this example is $\omega^\star = 3$.

Example2.6: Consider $Q = R$ and $\tilde{F}_N : Q \times R \rightarrow [0,1]$ be a fuzzy norm specified by: $\widetilde{N}(\omega, \tau) = \frac{\tau}{\tau + \|\omega\|}$, $\omega \in Q$ and $\tau > 0$, where $\|\omega\|: R \rightarrow [0, \infty)$ and $\|\omega\| = |\omega|$.

Let $\widetilde{L} = \{2,3,4\}\ and\ \widetilde{D} = \{6,7,8,9,10\}$,

$$N_d(\widetilde{L}, \widetilde{D}, \tau) = sup\{\widetilde{N}(\omega - \rho, \tau) : \omega \in \widetilde{L}, \rho \in \widetilde{D}\} = \frac{\tau}{\tau + 2}$$

Also, define $\Lambda: \widetilde{L} \rightarrow \widetilde{D}$ by

$\Lambda(\omega) = \{6, \quad if\ \omega = 4\ \omega + 4, \quad otherwise$

Observe that $\widetilde{L}\circ(\tau) = \{4\}$ and $\widetilde{D}\circ(\tau) = \{6\}$,

$\Lambda(\widetilde{L}\circ(\tau)) \subseteq \widetilde{D}\circ(\tau)$.

Assume that

$\{\ \widetilde{N}(u - \Lambda\omega, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)\ \widetilde{N}(v - \Lambda\rho, \tau) = N_d(\widetilde{L}, \widetilde{D}, \tau)$

Then $u = v = 4$.

Now, we define $\tilde{\varphi}, \tilde{\psi}:\ [0,\infty) \rightarrow [0,1]$ by $\tilde{\varphi}(\delta) = \delta$ and $\tilde{\psi}(\delta) = \delta^2$ for all $\delta \in [0,1]$ then we have

$$\tilde{N}(u-v,\tau) = \frac{\tau}{\tau+|4-4|} = 1 \geq \tilde{\varphi}\left(\tilde{N}(\omega-\rho,\tau)\right) + \tilde{\psi}(\tilde{N}(\omega-\rho,\tau)) \quad \text{for all } \tau > 0.$$

As a result, all of the requirements of Theorem 2.4 are fulfilled and the mapping Λ possesses unique ƁPP. The unique ƁPP of Λ in this example is $\omega^{\star} = 4$.

3. Conclusions

This paper introduces the notion of $\tilde{\varphi}-\tilde{\psi}-PC$ mapping in a fuzzy normed space and proves the ƁPP theorem for this mapping. To demonstrate the significance of the results, an example is provided. This work may be seen as the basis for future research that studies the applications of this kind of mapping within the framework of the notion of a fuzzy normed space.

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