

Fixed Point Results and Application of Cyclic Contractive Maps in b-Metric Spaces

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Abstract

One of the generalizations of the usual metric function is the b-metric, which provides researchers with a broader field for deriving numerous results and applications related to fixed point theory. The aim of this paper is to develop three new fixed point principles in the complete b-metric space (\mathcal{E}, ρ) when ρ is a continuous function in two variables. Here there are three directions to prove the existence and uniqueness of fixed points. First, we derive a result in terms of Branciari's theorem by combining integral contractive conditions with the notion of a cyclic map. Second, we apply the notion of cyclic representation to maps satisfying general weak conditions, including a changing distance function, to simulate the content of Boyd and Wong's theorem. Using this result, an application to the existence and uniqueness of the solution of an integral equation is given. Finally, an implicit relation with a changing distance function is used to construct a cyclic contractive map. Some examples are also presented to analyze and illustrate the main results.

Keywords: Alternating distance functions, Complete b-metric spaces, Contractive conditions, Cyclic representation, Fixed points.

1. Introduction

In 2003, Cyclic contraction was first introduced by Kirk et al. (1), who introduced results dealing with mappings of the type $f: \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$, $i = 1, 2, \dots, p + 1$, with $\mathcal{E}_i = \mathcal{E}_{i+1}$, where the contractive hypotheses are restricted to pairs $(a, b) \in \mathcal{E}_i \times \mathcal{E}_{i+1}$.

Extensions of the Banach's theorem and an extension of the Caristi theorem were proved. In addition, results related to non-expansive mappings in a Banach space were included. Several authors have contributed to research on fixed points in various cases for cyclic contractions (2-9). Backhtin (10) presented a definition of b-metric by replacing the triangle inequality as the following

Definition 1.1: Let \mathcal{E} be a nonempty set and $q \geq 1$. A function $\rho: \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}^+$ is said to be a b-metric on \mathcal{E} if (10):

1. $\rho(a, b) = 0$ if and only if $a = b$;

2. $\rho(a, b) = \rho(b, a)$ for all $a, b \in \mathcal{E}$;
3. $\rho(a, b) \leq q(\rho(a, c) + \rho(c, b))$ for all $a, b, c \in \mathcal{E}$.

The pair (\mathcal{E}, ρ) is called a b-metric space.

Definition 1.2: Let (\mathcal{E}, ρ) be a b-metric space, $a \in \mathcal{E}$ and (a_n) be a sequence in \mathcal{E} . Then

1. (a_n) converges to a if and only if $\lim_{n \rightarrow \infty} \rho(a_n, a) = 0$. We denote this by $\lim_{n \rightarrow \infty} a_n = a$ or $a_n \rightarrow a$ (as $n \rightarrow \infty$).
2. (a_n) is Cauchy if and only if $\lim_{n, m \rightarrow \infty} \rho(a_n, a_m) = 0$.
3. (\mathcal{E}, ρ) is complete if and only if every Cauchy sequence in \mathcal{E} is convergent.

Remark 1.3: In a b-metric space (\mathcal{E}, ρ) , the following assertions hold(11,12):

1. A convergent sequence has a unique limit.
2. Each convergent sequence is Cauchy.
3. In general, a b-metric is not continuous.

As in the usual metric space (1), we reform the following definition:

Definition1.4: Let $\{\mathcal{E}_i\}_{i=1}^n$ be a nonempty subsets of a b-metric space (\mathcal{E}, ρ) , $\mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$ and $f: \mathcal{E} \rightarrow \mathcal{E}$ such that

- 1) $f(\mathcal{E}_1) \subset \mathcal{E}_2, \dots, f(\mathcal{E}_{n-1}) \subset \mathcal{E}_n, f(\mathcal{E}_n) \subset \mathcal{E}_1$ for $1 \leq i \leq n$;
- 2) $\exists k \in (0,1)$ such that $\rho(fa, fb) \leq k \rho(a, b) \forall a \in \mathcal{E}_i, b \in \mathcal{E}_{i+1}$ for $1 \leq i \leq n$

Then f is the cyclic contraction map and \mathcal{E} is cyclic representation w.r.t., f .

In the field of fixed point theory for cyclicity, see (13-20)

Example1.5: Let $\mathcal{E} = [-1,1], \rho(a, b) = |a - b|^2$ is b-metric with $s = 2$ and $\mathcal{E}_1 = [-1,0], \mathcal{E}_2 = [0,1], \mathcal{E}_3 = [-1,0], \mathcal{E}_4 = [0,1], \mathcal{E}_5 = [-1,0], \mathcal{E}_6 = [0,1]$. So, $\mathcal{E} = \bigcup_{i=1}^6 \mathcal{E}_i$.

Define $f: \bigcup_{i=1}^6 \mathcal{E}_i \rightarrow \bigcup_{i=1}^6 \mathcal{E}_i$ such that $fa = -\frac{a}{2+a}, \forall a \in \bigcup_{i=1}^6 \mathcal{E}_i$. Here \mathcal{E} is cyclic representation w.r.t., f . ($f(\mathcal{E}_1) \subset \mathcal{E}_2, f(\mathcal{E}_2) \subset \mathcal{E}_3, f(\mathcal{E}_3) \subset \mathcal{E}_4$, and $f(\mathcal{E}_4) \subset \mathcal{E}_5, f(\mathcal{E}_5) \subset \mathcal{E}_6, f(\mathcal{E}_6) \subset \mathcal{E}_1$) and f is cyclic contraction with constant $0 < k = \frac{1}{2}$, where $a \in \mathcal{E}_i, b \in \mathcal{E}_{i+1}$.

Definition 1.6: A point a is called a fixed point of a map $f: \mathcal{E} \rightarrow \mathcal{E}$ if $f(a) = a$ (21).

An example, $a = 0$ is a fixed point of f in the previous example.

This work includes four main fixed point theorems based on different cyclic contractive conditions. Here, (\mathcal{E}, ρ) denote to complete b-metric space where ρ is continuous.

2. Materials and Methods

In this section, there are three axes; the first one is

Fixed point for cyclic contractive maps with integral condition

Not that, A Lebesgue-integrable function $\Psi: [0,1) \rightarrow [0,1)$ is called summable if $\int \Psi(r)dr < \infty$ (22).

Theorem 2.1: Let (\mathcal{E}, ρ) be b-metric space, $k \in (0,1)$, and let $f: \mathcal{E} \rightarrow \mathcal{E}$ be a map such that for each $a, b \in \mathcal{E}$

$$\int_0^{\rho(fa, fb)} \Psi(r) dr \leq k \int_0^{\rho(a, b)} \Psi(r) dr, \forall a \in \mathcal{E}_i, b \in \mathcal{E}_{i+1} \quad (1)$$

where, Ψ is summable on each compact subset of $[0, \infty)$, nonnegative and for any $\varepsilon > 0, \int_0^\varepsilon \Psi(r) dr > 0$. Then $\exists! c \in \bigcap_{i=1}^n \mathcal{E}_i$, moreover, $\forall a \in \mathcal{E}, \lim_{m \rightarrow \infty} f^m(a) = c$.

Proof: Let $a_0 \in \mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$ and consider $a_{m+1} = fa_m$ for each $m \in \mathbb{N} \cup \{0\}$, so for any $m \in \mathbb{N} \cup \{0\}$, $\exists i_m \in \{1, 2, \dots, n\}$ such that $a_m \in \mathcal{E}_{i_m}$ and $a_{m+1} \in \mathcal{E}_{i_m+1}$.

If $a_{m_0} = a_{m_0+1}$ for some m_0 then, since $a_{m_0+1} = fa_{m_0} = a_{m_0}$ this mean a_{m_0} is a fixed point

of f . Thus, assume that $a_m \neq a_{m+1}$ for all $m \in N \cup \{0\}$. So,

$$\int_0^{\rho(fa_m, fa_{m+1})} \Psi(r) dr \leq k \int_0^{\rho(a_m, a_{m+1})} \Psi(r) dr. \quad (2)$$

By repeating the inequality (2) m times, it follows directly

$$\int_0^{\rho(fa_m, fa_{m+1})} \Psi(r) dr \leq k \int_0^{\rho(a_m, a_{m+1})} \Psi(r) dr = k^m \int_0^{\rho(a_0, fa_0)} \Psi(r) dr.$$

As consequence, since $k \in (0,1)$, obtaining a monotone decreasing sequence

$$\left(\int_0^{\rho(fa_m, fa_{m+1})} \Psi(r) dr \right) = \left(\int_0^{\rho(a_{m+1}, a_{m+2})} \Psi(r) dr \right) \text{ which has lower bound is } 0. \text{ We have}$$

$$\rho(fa_m, fa_{m+1}) \rightarrow 0 \text{ as } m \rightarrow \infty. \text{ By properties of real sequence, } \left(\int_0^{\rho(fa_m, fa_{m+1})} \Psi(r) dr \right)$$

converges $\varepsilon \geq 0$ such that $\lim_{m \rightarrow \infty} \int_0^{\rho(a_{m+1}, a_{m+2})} \Psi(r) dr = \varepsilon$. Suppose that $\varepsilon > 0$, it is

enough to assume $\lim_{m \rightarrow \infty} \sup \rho(a_{m+1}, a_{m+2}) = \varepsilon > 0$. Then there exists a $u_\varepsilon \in N$ and a sequence

$$(fa_{m_u})_{u \geq u_\varepsilon} \text{ such that } \rho(fa_{m_u}, fa_{m_u+1}) \rightarrow \varepsilon > 0 \text{ as } u \rightarrow \infty \text{ and } \rho(fa_{m_u}, fa_{m_u+1}) \geq \frac{\varepsilon}{2}.$$

For each $u \geq u_\varepsilon$, by

$$\frac{\varepsilon}{2} \leq \lim_{m \rightarrow \infty} \rho(fa_{m_u}, fa_{m_u+1}) = \lim_{m \rightarrow \infty} \sup \rho(fa_{m_u}, fa_{m_u+1}) = \varepsilon > 0,$$

this is true only if $\varepsilon = 0$. The next step is proving that for each $a_0 \in E$, (a_m) is a Cauchy sequence. For $j > n$ define $E_j = E_i$ if $j = i \pmod n$.

Claim I: for all $\varepsilon > 0$ there exist $m \in N$ such that for all $j, i \geq m$, $j - i \equiv 1 \pmod n$ then $\rho(a_j, a_i) < \varepsilon$.

Suppose that there exists $\varepsilon > 0$ such that for each $m \in N$, one can find $j > i > m$ with $j - i \equiv 1 \pmod n$ satisfying $\rho(a_j, a_i) \geq \varepsilon$. Clearly, $\rho(a_m, a_{m+1}) < \varepsilon$.

Now, take $m \geq 2 \pmod n$. Then, corresponding to $i \geq m$ use can choose j in such a way that it is the smallest integer with $j > i$ satisfying $j - i \equiv 1 \pmod n$ and $\rho(a_j, a_i) \geq \varepsilon$. Therefore, $\rho(a_{j-n}, a_i) \leq \varepsilon$. By triangular inequality

$$\begin{aligned} \varepsilon &\leq \rho(a_j, a_i) \leq q(\rho(a_j, a_{i-n}) + \sum_{k=1}^n \rho(a_{i-k}, a_{i-k+1})) \\ &\leq q(\sum_{k=1}^n \rho(a_{i-k}, a_{i-k+1})) + q\varepsilon \rightarrow q\varepsilon \text{ as } n \rightarrow \infty. \end{aligned}$$

Again, by triangular inequality

$$\varepsilon \leq \rho(a_j, a_i) \leq q\rho(a_j, a_{j+1}) + q\rho(a_{j+1}, a_{i+1}) + q\rho(a_{i+1}, a_i) \rightarrow q\varepsilon \text{ as } j, i \rightarrow \infty,$$

we get

$$\int_0^{\rho(fa_{j+1}, fa_{i+1})} \Psi(r) dr \leq k \int_0^{\rho(a_{j+1}, a_{i+1})} \Psi(r) dr. \quad (3)$$

Letting $j, i \rightarrow \infty$ implies to $\int_0^{q\varepsilon} \Psi(r) dr \leq k \int_0^{q\varepsilon} \Psi(r) dr$, which is a contradiction. Therefore, the (Claim I) is proved.

Now, to prove (a_m) is Cauchy sequence in (E, ρ) . Fix $\varepsilon > 0$. By the claim, $\exists m_0$ such that if $j, i \geq m_0$ with $j - i \equiv 1 \pmod n$ $\rho(a_j, a_i) \leq \frac{\varepsilon}{n}$.

Since $\lim_{m \rightarrow \infty} \rho(a_m, a_{m+1}) = 0$, $\exists m_1 \in N$ such that $\rho(a_m, a_{m+1}) \leq \frac{\varepsilon}{n}$, $\forall m \geq m_1$

Suppose $c, v \geq \max\{m_0, m_1\}$ and $c > v$. Then there exists $h \in \{1, 2, \dots, n\}$ such that $v - c \equiv h \pmod n$. Therefore, $v - c + r \equiv 1 \pmod n$ for $r = n - h + 1$. So, getting

$$\begin{aligned} \rho(a_c, a_v) &\leq q(\rho(a_c, a_{v+r}) + \rho(a_{v+r}, a_v)). \\ &\leq q\rho(a_c, a_{v+r}) + q^2\rho(a_{v+r}, a_{v+r-1}) + q^3\rho(a_{v+r-1}, a_{v+r-2}) + \dots + q^r\rho(a_{v+1}, a_v) \end{aligned}$$

By $\rho(a_j, a_i) \leq \frac{\varepsilon}{n}$ and $\rho(a_m, a_{m+1}) \leq \frac{\varepsilon}{n}$ and from the last inequality,

$$\rho(a_c, a_v) \leq q\frac{\varepsilon}{n} + q^2\frac{\varepsilon}{n} + q^3\frac{\varepsilon}{n} + \dots + q^r\frac{\varepsilon}{n} \leq q\frac{\varepsilon}{n} \left(\frac{1}{1-q} \right) \rightarrow 0, \text{ as } n \rightarrow \infty$$

This proves that (a_m) is the Cauchy sequence. The completeness of (E, ρ) implies to exist $c \in$

\mathcal{E} such that $\lim_{m \rightarrow \infty} a_m = c$.

To prove c is a fixed point for f . Since $\mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$ is a cyclic representation of \mathcal{E} w.r.t., f , the sequence (a_m) has infinite terms in each \mathcal{E}_{i_m} for $i_m \in \{1, 2, \dots, n\}$. Closeness of \mathcal{E}_{i_m} for $i_m \in \{1, 2, \dots, n\}$ implies to $c \in \bigcap_{i=1}^n \mathcal{E}_i$. Suppose that $c \in \mathcal{E}_i$ and $fc \in \mathcal{E}_{i+1}$. Since (\mathcal{E}, ρ) is complete, there exists a point $c \in \mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$ such that $c = \lim_{m \rightarrow \infty} f a_m$

$$0 < \rho(c, fc) \leq q \rho(c, a_{m+1}) + q \rho(f a_m, fc) \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Indeed both $\rho(c, f a_{m_r})$ and $\rho(f a_{m_r}, fc)$ converge to 0 as $m \rightarrow \infty$, for the first one it is obvious, while for the second one we have

$$\int_0^{\rho(f a_m, fc)} \Psi(r) dr \leq k \int_0^{\rho(a_m, c)} \Psi(r) dr \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Now, if $\rho(f a_m, fc)$ does not converge to 0 as $m \rightarrow \infty$, then there exists a subsequence $(a_{m_r})_{r \in \mathbb{N}}$ of (a_m) with $a_{m_r} \in \mathcal{E}_{i-1}$ such that $\rho(f a_{m_r}, fc) \geq \varepsilon$ for a certain $\varepsilon > 0$, we have the following contradiction $0 < \int_0^\varepsilon \Psi(r) dr \leq \int_0^{\rho(f a_{m_r}, fc)} \Psi(r) dr \rightarrow 0$ as $r \rightarrow \infty$. This means that $\rho(c, fc) \leq 0$ thus, $\rho(c, fc) = 0$, c is the fixed point of f .

For the uniqueness of fixed point c . Assume a fixed point w of f differs from c , i.e., $fw = w$. The cyclic character of f and $c, w \in \mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$ are fixed points of f implying that $c, w \in \bigcap_{i=1}^n \mathcal{E}_i$. By (1), we obtain

$$\int_0^{\rho(c, w)} \Psi(r) dr = \int_0^{\rho(fc, fw)} \Psi(r) dr \leq k \int_0^{\rho(c, w)} \Psi(r) dr,$$

which is contradiction, consequently, $c = w$ for each $a \in \mathcal{E}$, $\lim_{m \rightarrow \infty} f^m a = c$.

Example 2.2 : Let $\mathcal{E} = [-1, 1]$ and $\rho(a, b) = |a - b|^2$ is b-metric with $q = 2$. Suppose $\mathcal{E}_1 = [-1, 0]$, $\mathcal{E}_2 = [0, 1]$ and $\mathcal{E} = \bigcup_{i=1}^2 \mathcal{E}_i$. Define $f: \bigcup_{i=1}^2 \mathcal{E}_i \rightarrow \bigcup_{i=1}^2 \mathcal{E}_i$ such that $(a) = \frac{-a}{2} \forall a$. So, $f(\mathcal{E}_1) \subset \mathcal{E}_2$, $f(\mathcal{E}_2) \subset \mathcal{E}_1$ and f is contraction of integral type with constant $k = \frac{1}{2} \in (0, 1)$ and $\Psi(r) = \frac{r}{3}$, for $a \in \mathcal{E}_1, b \in \mathcal{E}_2$

$$\int_0^{\rho(fa, fb)} \frac{r}{3} dr = \int_0^{\frac{1}{2}|b-a|^2} \frac{r}{3} dr \leq \frac{1}{2} \int_0^{\rho(a, b)} \frac{r}{3} dr.$$

Hence f satisfies the hypothesis of (Theorem 2.1), which has a unique fixed point at 0.

Fixed point for general cyclic($\emptyset - \psi$) weak contractive maps

Recall the following two definitions:

Definition 2.3: Let ψ is function where $\psi: [0, \infty) \rightarrow [0, \infty)$ is called altering distance function if satisfies(23).

1. ψ is monotone increasing and lower semi-continuous;
2. $\psi(r) = 0$ if and only if $r = 0$.

As in usual metric spaces (4), below, we reform many concepts in a b-metric space

Definition 2.4: Let (\mathcal{E}, ρ) be a b-metric space, n a positive integer $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n$ nonempty closed subsets of \mathcal{E} and $\mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$. An operator $f: \mathcal{E} \rightarrow \mathcal{E}$ is said to be a cyclic weak ($\emptyset - \psi$)-contraction if

- 1) $\mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$ is a cyclic representation of \mathcal{E} w.r.t., f .
- 2) $\emptyset(\rho(fa, fb)) \leq \emptyset(\rho(a, b)) - \psi(\rho(a, b))$, for any $a \in \mathcal{E}_i, b \in \mathcal{E}_{i+1}, i = 1, 2, \dots, n$, where $\mathcal{E}_{n+1} = \mathcal{E}_1$ and $\emptyset, \psi: [0, \infty) \rightarrow [0, \infty)$ is a non-decreasing and continuous function satisfying $\emptyset(r) > 0, \psi(r) > 0$ for $r \in (0, \infty)$ and $\emptyset(0) = 0, \psi(0) = 0$.

Theorem 2.5: Let f be a self-map of (\mathcal{E}, ρ) satisfies

$$\emptyset(\rho(fa, fb)) \leq \emptyset(M(a, b)) - \psi(N(a, b)), \forall a \in \mathcal{E}_i, b \in \mathcal{E}_{i+1} \quad (4)$$

where

$$M(a, b) = t \max\{\rho(a, b), \rho(a, fa), \rho(b, fb), \rho(a, fb), \rho(b, fa)\},$$

and

$$N(a, b) = t \min\{\rho(a, b), \rho(a, fa), \rho(b, fb), \rho(a, fb), \rho(b, fa)\},$$

where $t \in (0, 1)$, and $\psi, \phi: [0, \infty) \rightarrow [0, \infty)$ are altering distance functions. Then $\exists c \in \bigcap_{i=1}^n E_i$, c is a unique fixed point of f .

Proof: Let $a_0 \in \bigcup_{i=1}^n E_i$ and consider $a_{m+1} = fa_m$ for each $m \in N \cup \{0\}$, so for any $m \in N \cup \{0\}$, $\exists i_m \in \{1, 2, \dots, n\}$ such that $a_m \in E_{i_m}$ and $a_{m+1} \in E_{i_{m+1}}$.

If $a_{m_0} = a_{m_0+1}$ for some m_0 then, since $a_{m_0+1} = f^{m_0} a_0 = a_{m_0}$ this means a_{m_0} is fixed point of f . Thus, assume that $a_m \neq a_{m+1}$ for all $m \in N \cup \{0\}$. getting

$$\phi(\rho(a_m, a_{m+1})) \leq \phi(M(a_{m-1}, a_m)) - \psi(N(a_{m-1}, a_m)), \quad (5)$$

where

$$M(a_{m-1}, a_m) = t \max\{\rho(a_{m-1}, a_m), \rho(a_m, a_{m+1}), \rho(a_{m-1}, a_{m+1}), \rho(a_m, a_m)\},$$

and

$$N(a_{m-1}, a_m) = t \min\{\rho(a_{m-1}, a_m), \rho(a_m, a_{m+1}), \rho(a_{m-1}, a_{m+1}), 0\} = 0.$$

By triangular inequality, so

$$M(a_{m-1}, a_m) = t \max\{\rho(a_{m-1}, a_m), \rho(a_m, a_{m+1}), q\rho(a_{m-1}, a_m) + q\rho(a_m, a_{m+1}), 0\},$$

if $\rho(a_{m-1}, a_m) < \rho(a_m, a_{m+1})$, then $M(a_{m-1}, a_m) \leq t 2q\rho(a_m, a_{m+1})$. And by (5), we obtain $\phi(\rho(a_m, a_{m+1})) \leq \phi(t 2q\rho(a_m, a_{m+1}))$. Since ϕ is altering distance function, subsequently $\rho(a_m, a_{m+1}) \leq t 2q\rho(a_m, a_{m+1})$, this not true for all $t \in (0, 1)$, as result

$$M(a_{m-1}, a_m) = t 2q\rho(a_{m-1}, a_m) \text{ and } N(a_{m-1}, a_m) = 0. \quad (6)$$

Now put (6) in (5)

$$\phi(\rho(a_m, a_{m+1})) \leq \phi(t 2q\rho(a_{m-1}, a_m)). \quad (7)$$

Since ϕ is altering distance function, as a result $\rho(a_m, a_{m+1}) \leq t 2q\rho(a_{m-1}, a_m)$. Thus for all $m \in N \cup \{0\}$ we have a monotone decreasing sequence $(\rho(a_m, a_{m+1})) = (\rho(fa_{m-1}, fa_m))$. By properties of real sequence, $(\rho(a_m, a_{m+1}))$ there exist $\varepsilon \geq 0$ such that $\lim_{m \rightarrow \infty} \rho(a_m, a_{m+1}) = \varepsilon$. On letting $m \rightarrow \infty$ in (7), obtaining $\phi(\varepsilon) \leq \phi(t 2q\varepsilon)$. Assume that $\varepsilon \neq 0$. So since ϕ is altering distance function, getting $\varepsilon \leq t 2q\varepsilon < \varepsilon$, which a contradiction. Hence $\lim_{m \rightarrow \infty} \rho(a_m, a_{m+1}) = 0$.

For $j > n$ define $E_j = E_i$ if $j = i \pmod n$.

Claim I: for all $\varepsilon > 0$ there exist $m \in N$ such that for all $j, i \geq m$, $j - i \equiv 1 \pmod n$ then $\rho(a_j, a_i) < \varepsilon$.

Suppose that there exists $\varepsilon > 0$ such that for each $m \in N$ we can find $j > i > m$ with $j - i \equiv 1 \pmod n$ satisfying $\rho(a_i, a_j) \geq \varepsilon$.

Now, take $m \geq 2 \pmod n$. Then, corresponding to $i \geq m$ use can choose j in such a way that it is the smallest integer with $j > i$ satisfying $j - i \equiv 1 \pmod n$ and $\rho(a_j, a_i) \geq \varepsilon$. Therefore, $\rho(a_{j-n}, a_i) \leq \varepsilon$. By triangular inequality

$$\varepsilon \leq \rho(a_j, a_i) \leq q\rho(a_j, a_{j-n}) + q \sum_{k=1}^n \rho(a_{i-k}, a_{i-k+1}) \leq q \sum_{k=1}^n \rho(a_{i-k}, a_{i-k+1}) + q\varepsilon.$$

Taking $n \rightarrow \infty$ and since $\lim_{m \rightarrow \infty} \rho(a_m, a_{m+1}) = 0$, we obtain $\lim_{i, j \rightarrow \infty} \rho(a_j, a_i) = q\varepsilon$.

Again, by triangular inequality

$$\varepsilon \leq \rho(a_j, a_i) \leq 2q\rho(a_{j+1}, a_j) + q\rho(a_j, a_i) + 2q\rho(a_{i+1}, a_i).$$

Letting $j, i \rightarrow \infty$ and since $\lim_{m \rightarrow \infty} \rho(a_m, a_{m+1}) = 0$, so $\lim_{i, j \rightarrow \infty} \rho(a_{j+1}, a_{i+1}) = q\varepsilon$.

Since a_i, a_j belong to different sets E_i and E_{i+1} , and using (2.4)

$$\emptyset(\rho(fa_j, fa_i)) \leq \emptyset(M(a_j, a_i)) - \psi(N(a_j, a_i)), \quad (8)$$

where

$$M(a_j, a_i) = t \max\{\rho(a_j, a_i), \rho(a_j, a_{j+1}), \rho(a_i, a_{i+1}), \rho(a_j, a_{i+1}), \rho(a_i, a_{j+1})\}$$

$$N(a_j, a_i) = t \min\{\rho(a_j, a_i), \rho(a_j, a_{j+1}), \rho(a_i, a_{i+1}), \rho(a_j, a_{i+1}), \rho(a_i, a_{j+1})\}.$$

Since $\lim_{i,j \rightarrow \infty} \rho(a_j, a_i) = q\varepsilon$, $\lim_{i,j \rightarrow \infty} \rho(a_{j+1}, a_{i+1}) = q\varepsilon$, and by the triangle inequality

$$\rho(a_{i+1}, a_{j+1}) \leq q\rho(a_{i+1}, a_i) + q\rho(a_i, a_j) + q\rho(a_j, a_{j+1}) \rightarrow q\varepsilon \text{ as } i, j \rightarrow \infty.$$

Again by the triangle inequality

$$\rho(a_j, a_{i+1}) \leq q\rho(a_j, a_{j+1}) + q\rho(a_{j+1}, a_{i+1}) \rightarrow q\varepsilon \text{ as } i, j \rightarrow \infty.$$

Again by the triangle inequality

$$\rho(a_i, a_{j+1}) \leq q\rho(a_i, a_{i+1}) + q\rho(a_{i+1}, a_{j+1}) \rightarrow q\varepsilon \text{ as } i, j \rightarrow \infty.$$

Letting $i, j \rightarrow \infty$, we have $M(a_j, a_i) = t \max\{q\varepsilon, 0, 0, q\varepsilon, q\varepsilon\} \rightarrow tq\varepsilon$ and $N(a_j, a_i) \rightarrow 0$ and by the inequality (8), we have $\emptyset(\varepsilon) \leq \emptyset(tq\varepsilon) - \psi(0)$.

Since \emptyset is altering distance function and $t \in (0,1)$, then $\varepsilon \leq tq\varepsilon$ which a contradiction. Therefore, the claim (I) is held.

Now, we will prove (a_m) is a Cauchy sequence in (E, ρ) . Fix $\varepsilon > 0$. By the claim, $\exists m_0$ such that if $j, i \geq m_0$ with $j - i \equiv 1(\text{mod } n)$ such that $\rho(a_j, a_i) \leq \frac{\varepsilon}{2}$.

Since $\lim_{m \rightarrow \infty} \rho(a_m, a_{m+1}) = 0$, also $\exists m_1 \in N$ such that $\rho(a_m, a_{m+1}) \leq \frac{\varepsilon}{2n}$, $\forall m \geq m_1$.

Suppose $c, v \geq \max\{m_0, m_1\}$ and $c > v$. Then there exists $h \in \{1, 2, \dots, n\}$ such that $v - c \equiv h(\text{mod } n)$. Therefore, $v - c + r \equiv 1(\text{mod } n)$ for $r = n - h + 1$. So, we have

$$\rho(a_c, a_v) \leq q\rho(a_c, a_{v+r}) + q^2\rho(a_{v+r}, a_{v+r-1}) + q^3\rho(a_{v+r-1}, a_{v+r-2}) + \dots + q^r\rho(a_{v+1}, a_v) \quad (9)$$

By $\rho(a_j, a_i) \leq \frac{\varepsilon}{2}$ and $\rho(a_m, a_{m+1}) \leq \frac{\varepsilon}{2n}$ and from (9), $\rho(a_c, a_v) \leq q\frac{\varepsilon}{n}(\frac{1}{1-q}) \rightarrow 0$, as $n \rightarrow \infty$

This proves that (a_m) is a Cauchy sequence. The completeness of (E, ρ) implies to exists $c \in E$ such that $\lim_{m \rightarrow \infty} a_m = c$.

Now, to prove c is a fixed point for f . Since $E = \bigcup_{i=1}^n E_i$ is a cyclic representation of E w.r.t., f , the sequence (a_m) has infinite terms in each E_{i_m} for $i_m \in \{1, 2, \dots, n\}$. Closeness of E_{i_m} for $i_m \in \{1, 2, \dots, n\}$ implies to $c \in \bigcap_{i=1}^n E_i$. Suppose that $c \in E_i$ and $fc \in E_{i+1}$ and take a subsequence $(a_{m_r})_{r \in N}$ of (a_m) with $a_{m_r} \in E_{i-1}$ and take $a = c, b = a_{m_r}$ in (4)

$$\emptyset(\rho(fc, fa_{m_r})) \leq \emptyset(M(a_{m_r}, c)) - \psi(N(a_{m_r}, c)) \quad (10)$$

where

$$M(a_{m_r}, c) = t \max\{\rho(a_{m_r}, c), \rho(a_{m_r}, fa_{m_r}), \rho(c, fc), \rho(a_{m_r}, fc), \rho(c, fa_{m_r})\},$$

$$N(a_{m_r}, c) = t \min\{\rho(a_{m_r}, c), \rho(a_{m_r}, fa_{m_r}), \rho(c, fc), \rho(a_{m_r}, fc), \rho(c, fa_{m_r})\}.$$

Taking $r \rightarrow \infty$, hence

$$M(c, a_{m_r}) = t\rho(c, fc). \text{ And } (c, a_{m_r}) = 0, \text{ using (10) subsequently}$$

$$\emptyset(\rho(fc, c)) \leq \emptyset(t\rho(c, fc)) - \psi(0) \leq \emptyset(t\rho(c, fc)).$$

Since \emptyset is altering distance function, then $\rho(fc, c) < t\rho(c, fc)$, which a contradiction because $t \in (0,1)$, hence $fc = c$. Thus c is a fixed point of f .

For the uniqueness, suppose that there are two distinct points c, w with c and w fixed points of f . The cyclic character of f and the fact that $c, w \in E = \bigcup_{i=1}^n E_i$ are fixed points of f imply

that $c, w \in \bigcap_{i=1}^n E_i$. Using (4), we can obtain

$$\emptyset(\rho(fc, fw)) \leq \emptyset(M(c, w)) - \psi(N(c, w)),$$

Where, $\{M(c, w) = t \rho(c, w) \text{ and } N(c, w)\} = 0$. Hence $\emptyset(\rho(c, w)) = \emptyset(\rho(fc, fw)) \leq \emptyset(t \rho(c, w))$. And since \emptyset is altering distance function, we obtain $\rho(c, w) \leq q \rho(c, w)$ which is a contradiction since it is not true for all $t \in (0, 1)$, then $c = w$. Hence f has a unique fixed point in E and $c \in \bigcap_{i=1}^n E_i$.

Example 2.6: Let $E = \{6, 7, 8, 9, 10\}$ with $\rho: E \times E \rightarrow [0, \infty)$ defined by

$$\rho(a, b) = \begin{cases} 0, & \text{if } a = b \\ 6, & \text{if } a \neq b, a, b \in \{6, 7, 8, 9\} \\ 17, & \text{if } a, b \in \{9, 10\} \text{ and } a \neq b \\ 40, & \text{if } a \in \{6, 7, 8\} \text{ and } b = 10 \text{ (or } b \in \{6, 7, 8\} \text{ and } a = 10) \end{cases}.$$

Since all Cauchy sequences in E are constant. Therefore, are convergent. Then (E, ρ) is complete b-metric space with $q = 2$. And $E_1 = \{6, 8, 10\}$ and $E_2 = \{6, 7, 9\}$, $E = \bigcup_{i=1}^2 E_i$. Define $f: \bigcup_{i=1}^2 E_i \rightarrow \bigcup_{i=1}^2 E_i$ such that $f(a) = 6$ and if $a \in \{6, 7, 8, 9\}$ and $f(10) = 8$. So, $(E_1) \subset E_2, f(E_2) \subset E_1$, for $a \in E_1, b \in E_2$, and take $t = \frac{1}{4}$. Let $\emptyset, \psi: [0, \infty) \rightarrow [0, \infty)$ such that $\emptyset(r) = \frac{r}{4}$ and $\psi(r) = \frac{r}{2}$. Then \emptyset and ψ are altering distance functions. It is easy to check condition (4) holds with fixed point $a = 6$.

An application for solving integral equations. Consider the integral equation (24,25)

$$w(t) = \int_0^J Q(t, r)T(r, w(r)) dr \quad \text{for all } t \in [0, J], \quad (11)$$

where $J > 0$, $T: [0, J] \times R \rightarrow R$ and $Q: [0, J] \times [0, J] \rightarrow [0, \infty)$ are continuous functions.

In this section, we look for a nonnegative solution to (11) in $E = C([0, J], R)$ by (Theorem 2.5). Let $E = C[0, J]$ be the set of real valued continuous functions on $[0, J]$, where $[0, J]$ is a closed and bounded interval in R . For $p > 1$, define $\rho: [0, J] \times [0, J] \rightarrow R$ by

$$\rho(w, v) = \max_{t \in [0, J]} |w(t) - v(t)|^p, \text{ for all } w, v \in E.$$

Therefore, (E, ρ) is a complete b-metric space with $q = 2^{p-1}$. Let $\alpha, \beta \in E$ and $\alpha_0, \beta_0 \in R$ such that

$$\alpha_0 \leq \alpha(t) \leq \beta(t) \leq \beta_0, \quad \forall t \in [0, J]. \quad (12)$$

Suppose that for all $t \in [0, J]$, we have

$$\alpha(t) \leq \int_0^J Q(t, r)T(r, \beta(r)) dr, \quad (13)$$

and

$$\beta(t) \geq \int_0^J Q(t, r)T(r, \alpha(r)) dr. \quad (14)$$

We suppose that $\forall r \in [0, J], T(r, \cdot)$ be a decreasing function, that

$$a, b \in R, \quad a \geq b \text{ then } T(r, a) \leq T(r, b). \quad (15)$$

Assume that $k > 0$ is such that

$$k \left(\max_{t \in [0, J]} \int_0^J Q(t, r) dr \right) < 1. \quad (16)$$

Define a map $f: E \rightarrow E$ by $fw(t) = \int_0^J Q(t, r)T(r, w(r)) dr$, for all $t \in [0, J]$.

Suppose that $\forall r \in [0, J]$ and $a, b \in E$ with $(a(r) \leq \alpha_0 \text{ and } b(r) \leq \beta_0)$ or vice versa,

$$0 \leq [T(q, a(r)) - T(r, b(r))] \leq k \max\{|a(r) - b(r)|^p, 0, |b(r) - fb(r)|^p, |a(r) - fb(r)|^p, |b(r) - a(r)|^p\}^{\frac{1}{p}} \quad (17)$$

Theorem 2.7: Under the assumptions (12)-(17), the integral equation (11) has a solution in the set $\{w \in C([0, J]): \alpha \leq w \leq \beta\}$.

Proof : We omit the details because the proof steps are classic with some minor differences due to the specificity of the b-metric space.

Fixed points by implicit conditions

The following list of implicit functions under various conditions (26). Let Ω be the set of all real continuous functions $M: R_+^6 \rightarrow R$, satisfying the following conditions:

M_1) is non-increasing in variables r_2, r_3, r_4, r_5, r_6

M_2) there exists a right continuous function $A: [0, \infty) \rightarrow [0, \infty)$, $A(0) = 0, A(r) < r$, for $r > 0$ such that for $c \geq 0$ $M(c, u, c, u, 0, c+u) \leq 0$ or $M(c, u, 0, 0, u, u) \leq 0$, implies $c \leq A(u)$.

M_3) $M(c, 0, c, 0, 0, c) > 0$ and $M(c, c, 0, 0, c, c) > 0 \forall c > 0$.

Also, let $\Psi :=$ the set of functions $\psi: [0, \infty) \rightarrow [0, \infty)$ such that:

- i) ψ is monotone increasing and continuous;
- ii) $\psi(r) = 0$ if and only if $r = 0$;
- iii) ψ is subadditive, i.e., $\forall r_1, r_2 \in [0, +\infty), \psi(r_1 + r_2) = \psi(r_1) + \psi(r_2)$.

Lemma 2.8: Let $A: [0, \infty) \rightarrow [0, \infty)$ be a right continuous function such that $A(r) < r$, for $r > 0$. (27) . Then $\lim_{m \rightarrow \infty} A^m(r) = 0$, where $A^m := m$ times repeated composition of A .

Theorem 2.9: If $M \in \Omega$ exists and $E = \bigcup_{i=1}^n E_i$ is a cyclic representation of E w.r.t., $f: E \rightarrow E$.

If for any $(a, b) \in E_i \times E_{i+1}, i = 1, 2, \dots, n$

$$M(\psi(\rho(fa, fb)), \psi(\rho(a, b)), \psi(\rho(a, fa)), \psi(\rho(b, fb)), \psi(\rho(a, fb)), \psi(\rho(b, fa))) \leq 0$$

(18), and $\psi \in \Psi$. $\exists!$ $c \in \bigcap_{i=1}^n E_i$, c is a unique fixed point. Moreover, $\lim_{m \rightarrow \infty} f^m(a) = c$, for any $a \in E$.

Proof: Let $a_0 \in \bigcup_{i=1}^n E_i$ and a_m define by $a_{m+1} = fa_m$. So for $m \geq 0, \exists i_m \in \{1, 2, \dots, n\}$ such that $a_{m-1} \in E_{i_m}$ and $a_m \in E_{i_{m+1}}$. If $a_{m_0} = a_{m_0-1}$ for some $m_0, a_{m_0} = fa_{m_0-1} = a_{m_0-1}$ then a_{m_0} is fixed point of f . Thus, suppose that $a_m \neq a_{m-1}$, for all $m \in N \cup \{0\}$. By using (2.18) therefore

$$M(\psi(\rho(a_{m+1}, a_m)), \psi(\rho(a_m, a_{m-1})), \psi(\rho(a_m, a_{m+1})), \psi(\rho(a_{m-1}, a_m)), \psi(\rho(a_m, a_m)), \psi(\rho(a_{m-1}, a_{m+1}))) \leq 0.$$

And since $\psi(r) = 0$ if and only if $r = 0$, also by using triangle inequality and since ψ is subadditive, therefore

$$M(\psi(\rho(a_{m+1}, a_m)), \psi(\rho(a_m, a_{m-1})), \psi(\rho(a_m, a_{m+1})), \psi(\rho(a_{m-1}, a_m)), 0, \psi(q\rho(a_{m-1}, a_m)) + \psi(q\rho(a_m, a_{m+1}))) \leq 0.$$

And from M_2 , there exists a right continuous function $A: [0, \infty) \rightarrow [0, \infty)$, $A(0) = 0, A(r) < r$, for $r > 0$, such that for all $m \in N \cup \{0\}$

$$\psi(\rho(a_{m+1}, a_m)) \leq A(\psi(\rho(a_m, a_{m-1}))).$$

If we use this procedure, we get

$$\psi(\rho(a_{m+1}, a_m)) \leq A(\psi(\rho(a_m, a_{m-1}))) \leq \dots \leq A^m(\psi(\rho(a_1, a_0))). \quad (19)$$

And by (Lemma 8) and continuity of ψ , subsequently

$$\lim_{m \rightarrow \infty} \psi(\rho(a_{m+1}, a_m)) = 0 = \psi(\lim_{m \rightarrow \infty} \rho(a_{m+1}, a_m)). \text{ Since } \psi(r) = 0 \text{ if and only if } r = 0, \text{ therefore}$$

$$\lim_{m \rightarrow \infty} \rho(a_{m+1}, a_m) = 0. \quad (20)$$

To prove for each $a_0 \in E$, (a_m) is a Cauchy sequence. Assume it is false. Then we can find a $\varepsilon > 0$ and $\{p_r\}, \{d_r\}, d_r > p_r \geq r$ where $\{p_r\}, \{d_r\}$ two subsequences of integers with

$$\psi(\rho(a_{p_r}, a_{d_r})) \geq \varepsilon \text{ for } n \in \{1, 2, \dots\}. \quad (21)$$

We also assume

$$\psi(\rho(a_{p_r}, a_{d_r-1})) < \varepsilon. \quad (22)$$

By selecting d_r to be the least number surpassing p_r for which inequality (21) holds, now by (19) and (21), (22), and since ψ is subadditive, getting

$$\begin{aligned} \varepsilon &\leq \psi(\rho(a_{p_r}, a_{d_r-1})) \leq \psi(q\rho(a_{p_r}, a_{d_r-1})) + \psi(q\rho(a_{d_r-1}, a_{p_r})) \\ &\leq q\varepsilon + A^{d_r-1}\psi(q\rho(a_0, a_1)). \end{aligned} \quad (23)$$

And so

$$\lim_{r \rightarrow \infty} \psi(\rho(a_{d_r-1}, a_{p_r})) = q\varepsilon. \quad (24)$$

On the other hand, $\forall r, \exists i_r \in \{1, 2, \dots, n\}$ such that $d_r - p_r + i_r \equiv 1 \pmod{n}$. Then $a_{p_r-i_r}$ (for r large enough, $p_r > i_r$) and a_{d_r} belong to different sets E_i and E_{i+1} for $i \in \{1, 2, \dots, n\}$. By the triangle inequality, also ψ is subadditive, obtaining

$$\begin{aligned} \psi(\rho(a_{d_r}, a_{p_r})) &\leq \psi(q\rho(a_{d_r}, a_{d_r-i_r}) + q\rho(a_{d_r-i_r}, a_{p_r})) \\ &\leq \psi(q\rho(a_{d_r}, a_{d_r-i_r})) + \psi(q\rho(a_{d_r-i_r}, a_{p_r})). \end{aligned}$$

Now taking $r \rightarrow \infty$, by (19) and from (24), as a results

$$\psi(\rho(a_{p_r}, a_{d_r})) \leq A^{d_r-i_r}\psi(q\rho(a_0, a_1)) + q\varepsilon.$$

And so,

$$\lim_{r \rightarrow \infty} \psi(\rho(a_{d_r}, a_{p_r})) = q\varepsilon. \quad (25)$$

By using (20), so

$$\lim_{r \rightarrow \infty} \psi(\rho(a_{d_r+1}, a_{d_r})) = 0, \quad \lim_{r \rightarrow \infty} \psi(\rho(a_{p_r-i_r+1}, a_{p_r-i_r})) = 0. \quad (26)$$

And by using the triangle inequality, then

$$\psi(\rho(a_{p_r-i_r}, a_{d_r})) \leq \psi(q\rho(a_{p_r-i_r}, a_{p_r})) + \psi(q\rho(a_{p_r}, a_{d_r})).$$

Letting $r \rightarrow \infty$ in the last inequality and using (2.19) and (2.25), consequently

$$\begin{aligned} \psi(\rho(a_{p_r-i_r}, a_{d_r})) &\leq A^{p_r-i_r}\psi(q\rho(a_0, a_1)) + q\varepsilon \\ \lim_{r \rightarrow \infty} \psi(\rho(a_{p_r-i_r}, a_{d_r})) &= q\varepsilon. \end{aligned} \quad (27)$$

Again, by using the triangle inequality, obtaining

$$\psi(\rho(a_{p_r-i_r}, a_{d_r+1})) \leq \psi(q\rho(a_{p_r-i_r}, a_{d_r})) + \psi(q\rho(a_{d_r}, a_{d_r+1})).$$

Letting $r \rightarrow \infty$ in the last inequality and using (26) and (27), therefore

$$\lim_{r \rightarrow \infty} \psi(\rho(a_{p_r-i_r}, a_{d_r+1})) = q\varepsilon. \quad (28)$$

And in the same way, as a results

$$\psi(\rho(a_{d_r}, a_{p_r-i_r+1})) \leq \psi(q\rho(a_{d_r}, a_{p_r-i_r})) + \psi(q\rho(a_{p_r-i_r}, a_{p_r-i_r+1})).$$

Letting $r \rightarrow \infty$ and using (27) and (26), getting

$$\lim_{r \rightarrow \infty} \psi(\rho(a_{d_r}, a_{p_r-i_r+1})) = q\varepsilon. \quad (29)$$

Again, by using the triangle inequality, therefore

$$\psi(\rho(a_{p_r-i_r+1}, a_{d_r+1})) \leq \psi(q\rho(a_{p_r-i_r+1}, a_{d_r})) + \psi(q\rho(a_{d_r}, a_{d_r+1})).$$

Letting $r \rightarrow \infty$ in the last inequality and using (2.26) and (2.29), having

$$\lim_{r \rightarrow \infty} \psi(\rho(a_{p_r-i_r+1}, a_{d_r+1})) = q\varepsilon. \quad (30)$$

Using (2.18) for $a = a_{p_r-i_r}$ and $b = a_{d_r}$, subsequently

$$M(\psi(\rho(a_{p_r-i_r+1}, a_{d_r+1})), \psi(\rho(a_{p_r-i_r}, a_{d_r})), \psi(\rho(a_{p_r-i_r}, a_{p_r-i_r+1})),$$

$$\psi(\rho(a_{d_r}, a_{d_{r+1}})), \psi(\rho(a_{d_{r+1}}, a_{p_r-i_r})), \psi(\rho(a_{d_r}, a_{p_r-i_r+1}))) \leq 0.$$

Letting $r \rightarrow \infty$, and using (30), (27), (26), (28), (29), then, by continuity of M and $\psi(r) = 0$ if and only if $r = 0$, $M(q\varepsilon, q\varepsilon, 0, 0, q\varepsilon, q\varepsilon) \leq 0$, a contradiction with M_3 . Thus, (a_m) is Cauchy sequence in (\mathcal{E}, ρ) . Now to prove that c is fixed point of f . In fact $fa_m \rightarrow c$ and since $\mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$ is cyclic representation of \mathcal{E} w.r.t., f , the sequence (a_m) has infinite terms in each \mathcal{E}_{i_m} for $i_m \in \{1, 2, \dots, n\}$. Considering that \mathcal{E}_{i_m} is closed for $i_m \in \{1, 2, \dots, n\}$ we have $c \in \bigcap_{i=1}^n \mathcal{E}_i$.

Suppose that $c \in \mathcal{E}_i$ and $fc \in \mathcal{E}_{i+1}$, and take a subsequence $(a_{m_r})_{r \in \mathbb{N}}$ of (a_m) with $a_{m_r} \in \mathcal{E}_{i-1}$, using (2.18), take $a = c$ and $b = a_{m_r}$, as result

$$M(\psi(\rho(fc, fa_{m_r})), \psi(\rho(c, a_{m_r})), \psi(\rho(c, fc)), \psi(\rho(a_{m_r}, fa_{m_r})), \psi(\rho(c, fa_{m_r})), \psi(\rho(a_{m_r}, fc))) \leq 0.$$

Taking $r \rightarrow \infty$, hence

$$M(\psi(\rho(fc, c)), \psi(\rho(c, c)), \psi(\rho(c, fc)), \psi(\rho(c, c)), \psi(\rho(c, c)), \psi(\rho(c, fc))) \leq 0,$$

and $\psi(r) = 0$ if and only if $r = 0$, then $M(\psi(\rho(fc, c)), 0, \psi(\rho(c, fc)), 0, 0, \psi(\rho(c, fc))) \leq 0$, which is contradiction to M_3 . Thus $\psi(\rho(fc, c)) = 0$ then $\rho(fc, c) = 0$, then $fc = c$. We obtain c is a fixed point of f . Suppose that there are two distinct points c, w with c and w are two fixed points of f . The cyclic nature of f , also, the fact $c, w \in \mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$ are fixed points of f imply that $c, w \in \bigcap_{i=1}^n \mathcal{E}_i$ and by (18), obtaining

$$M(\psi(\rho(fc, fw)), \psi(\rho(c, w)), \psi(\rho(c, fc)), \psi(\rho(w, fw)), \psi(\rho(c, fw)), \psi(\rho(w, fc))) \leq 0.$$

Since c, w are fixed points of f , $\psi(r) = 0$ if and only if $r = 0$, we get

$$M(\psi(\rho(c, w)), \psi(\rho(c, w)), 0, 0, \psi(\rho(c, fw)), \psi(\rho(w, c))) \leq 0,$$

which is a contradiction to M_3 , then $\psi(\rho(c, w)) = 0$, hence $\rho(c, w) = 0$, that is, $c = w$. Hence f has a unique fixed point in \mathcal{E} and $c \in \bigcap_{i=1}^n \mathcal{E}_i$.

Remark 2.10: It is worth noting, it is worth noting that we can obtain good results by including the concept of cyclicity in cases of (28,29).

3. Discussion

This work is classified within the field depending on the classification 2010 MSC: 47H09, 47H10. Our study of this topic is the first in Iraq (to the best of our knowledge) in the field of fixed points for cyclic maps, and it is taken from a master's thesis by researcher Abbas Karim Nahi. The paper included new results in the field of integral contractions in the b-metric spaces of the cyclic type, as well as new generalizations of the results of other researchers in the case of the b-metric space. In the future, we would like to study the results in (30) in the case of cyclic maps.

4. Conclusions

In this paper, new theorems were established to find fixed points. First, by merging integral contractive conditions with the concept of cyclic map, and second, by applying the concept of cyclic representation with respect to maps satisfying general weak conditions, including a changing distance function. And third, by merging the changing distance function with the sub additive to find theorems in the field of fixed points through the concept of cyclic representation for maps satisfying an implicit relation including a changing distance function.

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Conflict of Interest

The authors declare no conflict of interest.

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