Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2026; 39(1)

Optimal Adomian Decomposition Method for Solving Nonlinear Ordinary Differential Equations in Sciences and Engineering

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Received: 25/April/2025
Accepted: 27/July/2025
Published:20/January/2026
doi.org/10.30526/39.1.4151

Abstract
This paper shows new approximate ways to solve nonlinear ordinary differential equations using two methods that repeat steps: the Adomian decomposition method (ADM) and the optimal Adomian decomposition method (OADM). These equations are extensively utilized in fluid dynamics and engineering. The OADM sets itself apart by incorporating an optimal control parameter that enhances solution accuracy and accelerates convergence, providing a distinct advantage over the ADM. The two methods have been applied to three important equations: the Darcy-Brinkmann-Forchheimer moment equation, the Blasius equation, and the Falkner-Skan equation. The effectiveness of the two methods was assessed by looking at how quickly they converged and the largest error remaining, while also comparing them to other numerical results from operational matrix methods found in existing research. The results demonstrate the superior accuracy of OADM, which proves its effectiveness in solving the nonlinear equations. All computations were conducted utilizing the program, which facilitated the execution and evaluation of the proposed methods.
Keywords: Adomian Decomposition Method; Optimal Adomian Decomposition Method; Maximum error remainder; Darcy-Brinkman-Forchheimer Moment Equation; Blasius Equation; Falkner-Skan Equation.

1.Introduction

Complex phenomena in science and engineering require accurate mathematical models. Nonlinear ordinary differential equations (NODEs) are essential for describing complex, changing behaviors, particularly in fluid flow and thermal expansion[1–3]. The ADM and OADM are effective for solving NODEs. It is based on the analysis of nonlinear components into polynomials called "Adomian polynomials", which facilitates the systematic and iterative construction of the solution. This method has received wide attention from researchers in recent years due to its accuracy and high efficiency in finding solutions[4,5]. Numerous research studies have employed ADM for solving many problems, including the Lane-Emden, and Riccati differential equations[6,7]. The convergence in this method has been given much attention by researchers; in 2009, studies of the convergence of the ADM with initial-value problems in the context of differential equations, as noted in[8]. In 2004, the ADM was extended to delay differential equations (DDE), where accurate approximate solutions were obtained using rapidly convergent series expansions, as in[9]. A recent study presented fourth- and fifth-order iterative schemes for solving coupled systems with nonlinear equations using the method of Adomian decomposition, as in[10]. While ADM is effective in addressing differential equations, studies indicate it may be ineffective in specific instances due to its sluggish convergence, resulting in imprecise or discontinuous sequential solutions[11]. Consequently, the necessity emerged to

enhance the convergence by adding an optimal parameter (a), and its value can be determined by the squared residual error[11]. A nonlinear iterative formula is established, and the procedure is reiterated until the requisite precision is attained. The method seeks to enhance the effectiveness by accelerating convergence and expanding the convergence region[11].
The Darcy–Brinkmann–Forchheimer moment equation (DBFME) is a fundamental equation for modeling fluid flow in porous media, garnering considerable attention from researchers in recent decades; see[12]. In 2006, it was demonstrated that the DBFME can be effectively treated using asymptotic techniques to analyze forced convection in porous channels; refer to[13]. In 2021, researchers employed the optimal Galerkin-homotopy asymptotic method to solve the same equation; refer to[14]. Recently, numerous approximation methods have garnered heightened interest for the DBFME. Among them are the Bernoulli, Bernstein, and shifting Legendre operational matrix method; see[2].
The Blasius equation has received major attention in the research community due to its importance in analyzing the behavior of the hydrodynamic boundary layer and the flow of viscous fluids in fluid mechanics[3]. Later, ADM and the differential transform method were employed to obtain semi-analytical solutions; refer to[15]. Furthermore, three different techniques were utilized in other studies: the simple perturbation technique, the Galerkin method, and the direct numerical method, with the advantages of each of them being evaluated based on the nature of the chosen field; see[16]. In addition, she employed the Crocco-Wang transform along with adjusted finite differences and the Wynn algorithm to obtain accurate solutions to the Blasius equation; see[17].
Numerous numerical and approximation methods have considered the Falkner–Skan equation as an important model in boundary layer theory. The shooting technique, a numerical method for solving boundary value problems was employed; refer to[18]. The ADM was utilized in conjunction with the Padé approximation in 2008; see[19]. Recently, A hybrid method integrating the Jaya algorithm with the Runge–Kutta method has been employed; see[20]. Additionally, operational matrix methods facilitated the acquisition of effective approximate solutions; refer to[1].
This paper is structured as follows: Section 2 addresses the essential equations of the proposed applications. Section 3 illustrates the proposed iterative methods, while Section 4 presents the convergence analysis along with approximate solutions and numerical results obtained using the proposed methods, whereas the last section discusses the conclusions drawn from this paper.

2. The Formulation of Some Applications

2.1. The Darcy-Brinkman-Forchheimer Moment Equation

The Darcy-Brinkmann-Forchheimer equation is a classic example of the boundary value problem in the analysis of fluid dynamics within porous media, since these equations appear in various physical, biological, and applied sciences problems[2]. The use of porous media in contemporary technology is increasing, including effects on thermal insulation, direct heat exchangers, and nuclear waste repositories[21]. It seemed necessary to determine solutions to these equations using numerical or approximate methods[2]. The classical Darcy's law became ineffective in the presence of inertia and rigid barriers, in particular at high Reynolds numbers determined by the pore size [13,21]. Consequently, the Darcy law was augmented to incorporate inertial and boundary effects with the introduction of the Brinkman and Forchheimer terms, thus enabling a more precise characterization of fluid flow in porous media. The results suggest that the addition of these effects can improve the thermal efficiency of heat exchangers[21]. The addition of these effects resulted in the formation of a second-order nonlinear differential equation with boundary conditions defined by the following formula[22].

$$u''(t) - s^2\, u(t) - F\, s\, u^2(t) + \frac{1}{M} = 0, \quad u'(0) = 0, u(1) = 0. \tag{1}$$

where s denotes the shape parameter of the porous media, F indicates the Forchheimer number, and M represents the viscosity ratio.

2.2. The Blasius Equation

The flow showing how an incompressible fluid moves in two dimensions over a flat plate that continues almost infinitely is explained by the boundary layer equation represented by the following formula[23]:

$$[|u''(t)|^{\rho-1}u''(t)]' + \frac{1}{\rho+1}\, u(t)u''(t) = 0 \quad \textbf{(2)}$$

subject to the boundary constraints: $u(0) = u'(0) = 0,\ u'(\infty) = 1$ **(3)**

The research studies' results indicate that the function $u(t)$ exhibits linear behavior when the power law coefficient is $\rho > 1$. The Blasius equation is extensively utilized in the examination of boundary layers in Newtonian liquids[23-25].

$$u'''(t) + \frac{1}{2}\, u(t)\, u''(t) = 0 \quad \textbf{(4)}$$

subject to the boundary conditions: $u(0) = u'(0) = 0, u'(\infty) = 1$ **(5)**

This equation is of high importance in the field of fluid mechanics. Specifically, when studying the dynamical layer borders and stratified flows of fluids with viscosity. Liao found that $u''(0) = 0.3320573$, a value commonly used in scientific literature[3]. This value was derived utilizing precise numerical methods grounded on standard assumptions, and it will be adopted in this paper.

2.3. The Falkner-Skan Equation

The Falkner-Skan equation, a NODE of the third order, first appeared in 1931. The importance of this equation lies in fluid mechanics and boundary layer theory and is used in different fields, such as the formation of plastic panels and insulating materials, the study of polymers, and the examination of incompressible boundary layers in two-dimensional equations[1,18], which is represented by the following formula[19]:

$$u'''(t) + u(t)\, u''(t) + \theta[1 - \left(u'(t)\right)^2] = 0, 0 < t < \infty. \quad \textbf{(6)}$$

with the boundary conditions: $u(0) = 0, u'(0) = -\varepsilon, u'(\infty) = 1.$

This equation is a common example of a boundary value equation in infinite domains. The main problem is dealing with infinite boundary conditions. This issue is resolved by transforming the semi-infinite physical field into a static arithmetic field through sophisticated numerical methods[18]. The behavior of the Falkner-Skan equation is contingent upon θ and ε, where θ denotes the coefficient of expansion of the moving boundary and ε signifies the velocity ratio in the free stream. The numerical study has revealed that the values of these two parameters influence many solutions. For instance, three solutions emerge for θ values ranging from 0 to 0.14, a singular solution exists when θ is between 0.14 and 0.5, and ultimately, two solutions are present when θ is between 0.5 and 1, illustrating the complex behaviors of solutions under changing parameters[26]. In reference[1], the researchers established the initial condition $u''(0) = -0.832666$ utilizing the boundary condition $u'(\infty) = 1$ through the method of Pade$'$ approximation. The formula will be modified to align with this condition and will serve as the foundation for this paper.

$$u'''(t) + u(t)\, u''(t) + \theta[\varepsilon^2 - \left(u'(t)\right)^2] = 0, \quad \textbf{(7)}$$

with the boundary conditions: $u(0) = 0, u'(0) = 1 - \varepsilon, u''(0) = -0.832666$ **(8)**

3. The Adomian Decomposition Method and Optimal Adomian Decomposition Method

3.1. The Fundamental Idea of the ADM

The ADM, developed by George Adomian from the 1970s to the 1990s, is a semi-analytical method intended for the solution of non-linear ordinary and partial differential equations[27]. The method uses Adomian polynomials to analyze the nonlinear components of differential equations. The ADM is a fundamental method for solving nonlinear differential equations, offering an efficient framework for obtaining accurate yet fast solutions, which makes it of great importance in addressing many mathematical and applied problems[28].

Consider the general nonlinear differential equation[11]:

$$\ell(u) + \mathcal{N}(u) = f(t), b \leq t \leq c, \quad \textbf{(9)}$$

with the imposition of the initial or boundary conditions of the differential equation,

where ℓ denotes the higher-order linear differential operator, while $\mathcal{N}$ indicates the nonlinear differential operator. where u is the function to be solved, while $f(t)$ denotes a known analytical function[28]. If ℓ is the first-order operator defined by $\ell = \frac{d}{dt}$. Assuming that ℓ is invertible, then the inverse operator ℓ^{-1} is given by $\ell^{-1}(\cdot) = \int_0^t(\cdot)dr$. If ℓ is a second-order differential operator, then $\ell = \frac{d^2}{dt^2}$. The inverse operator ℓ^{-1} is a two-fold integration operator given by $\ell^{-1}(\cdot) = \int_0^t\int_0^t(\cdot)drdr$, and so on. Applying the inverse operator ℓ^{-1} on both sides of **Equation 9**, the following is obtained[27]:

$$u = g(t) - \ell^{-1}[\mathcal{N}(u)], \quad \textbf{(10)}$$

where $g(t)$ is depends on the initial or boundary conditions of the differential equation, as well as the integral of the function $f(t)$, a nonlinear operator $\mathcal{N}(u)$ is usually represented by an infinite series, and this series (A_k) is called an Adomian polynomial, which is denoted by the next formula:

$$\mathcal{N}(u) = \sum_{k=0}^{\infty} A_k\,, u(t) = \sum_{k=0}^{\infty} u_k\,(t), \text{ for } k \geq 0 \quad \textbf{(11)}$$

The components $u_k(t)$ are determined using an iterative process as follows:

$$u_0(t) = g(t) + \ell^{-1}[f(t)], \quad \textbf{(12)}$$

$$u_{k+1}(t) = -\ell^{-1}[\mathcal{N}(u)],$$

$$u_{k+1}(t) = -\ell^{-1}[\textstyle\sum_{k=0}^{\infty} A_k\,], \text{ for } k \geq 0 \quad \textbf{(13)}$$

Replacing these components provides an approximate solution of order (k):

$$u(t) = \sum_{n=0}^{k} u_n\,(t) \text{ or } u(t) = u_0(t) + u_1(t) + u_2(t) + \cdots + u_k(t), \quad \textbf{(14)}$$

A primary challenge in this method lies in the calculation of Adomian polynomials, which denote nonlinear terms. There are two methods for computing these terms: The first method, introduced by Adomian, provides direct formulas for calculating these limits, where $A_k = A_k\,(u_0, u_1, \cdots, u_k)$, represents the Adomian polynomials, which can be determined using the following formula[29]:

$$A_k = \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\,[\mathcal{N}(\textstyle\sum_{n=0}^{\infty} u_n\,\lambda^n)]|_{\lambda=0}, \text{ for } k \geq 0, \quad \textbf{(15)}$$

The A_k are generated for all types of nonlinearities so that A_0 depends only on u_0, A_1 depends on u_0 and u_1, and so on. The A_k can be expressed as follows[29]:

$$A_0 = \mathcal{N}(u_0),$$

$$A_1 = u_1\mathcal{N}'(u_0),$$

$$A_2 = u_2\mathcal{N}'(u_0) + \frac{1}{2}u_1^2\mathcal{N}''(u_0),$$

$$A_3 = u_3\mathcal{N}'(u_0) + u_1u_2\mathcal{N}''(u_0) + \frac{1}{3!}u_1^3\mathcal{N}'''(u_0), \quad \textbf{(16)}$$

$$\vdots$$

Although the original formulas presented by Adomian are a direct way to calculate nonlinear terms[29], the researchers sought to develop an alternative method that would be simpler and more flexible. This method relies on algebraic operations and trigonometric properties, in addition to Taylor series, to reconstruct polynomials without relying on the original complex formulations. The core idea of this method involves systematically decomposing the nonlinear terms, initially assuming that the unknown function $u(t)$ is represented as an infinite series, $u(t) = \sum_{k=0}^{\infty} u_k\,(t)$. The initial value of the Adomian polynomial is determined by the relation $A_0 = \mathcal{N}(u_0)$. Subsequently, the nonlinear polynomials are constructed such that the sum of the sub-indices of the components in each A_k equals k, thereby enabling each A_k to be distinctly categorized according to its order in the sequence.

In other words, in the case $\mathcal{N}(u) = u^2(t)$, it is calculated based on the following steps[30]:
Step one: representing $u(t)$ as an infinite string, $u(t) = \sum_{k=0}^{\infty} u_k\,(t)$.
Step two: compensate $u(t)$ in $\mathcal{N}(u) = u^2(t)$, $\mathcal{N}(u) = (u_0 + u_1 + u_2 + u_3 + \cdots)^2$.
Step three: expand the expression on the right-hand side,

$$\mathcal{N}(u) = \mathrm{u}_0^2 + 2u_0u_1 + \mathrm{u}_1^2 + 2u_0u_2 + 2u_1u_2 + 2u_0u_3 + \cdots, \tag{17}$$

Then the Adomian polynomials can be expressed as:

$$A_0 = u_0^2(t),$$
$$A_1 = 2u_0(t)u_1(t),$$
$$A_2 = u_1^2(t) + 2u_0(t)u_2(t),$$
$$A_3 = 2u_1(t)u_2(t) + 2u_0(t)u_3(t),$$
$$\vdots$$

This method makes the Adomian polynomials clear and easy to determine, eliminating the need for the complex formulas provided by Adomian. This paper will employ this method to calculate nonlinear terms in the suggested applications.

3.2. The Fundamental Idea of the OADM

This method builds upon the general form of the differential equation as presented in **Equation (9)**, with the imposition of initial or boundary conditions. This method proposes an iterative scheme that introduces the optimal parameter a to enhance the convergence speed. The initial solution $u_0(t,a)$ is defined as follows[11]:

$$u_0(t,a) = g(t) - at, \tag{18}$$
$$u_1(t,a) = at + \ell^{-1}(g(t)), \tag{19}$$
$$u_{k+1}(t,a) = -\ell^{-1}[A_k(t)], \text{ for } k \geq 1. \tag{20}$$

Use for obtaining an approximate solution to **Equation 9**, which is dependent on the parameter a utilized to control the speed of convergence. Omitting this parameter may render the ADM incapable of achieving an accurate result. Utilizing Adamian polynomials, which denote nonlinear limits, and a parameter value a, an approximate solution of the equation can be found as follows:

$$u_k(t,a) = \sum_{n=0}^{k} u_n(t,a), \tag{21}$$

For obtaining the ideal value of parameter a, it is advisable to utilize the residual error equation, considering this value increases convergence speed and enhances results accuracy[11].
There are two methods for calculating the residual error: The first method is known as the analytical method, which consists of calculating the variance between approximate value and exact value, represented by the following mathematical formula:

$$Res(a) = \int_c^b [\ell(u_k(t)) + \mathcal{N}(u_k(t)) - f(t)]^2\, dt, \tag{22}$$

Determining the interval $[b,c]$ facilitates the minimization of this error.

$$\frac{dRes(a)}{da} = 0, \tag{23}$$

The second method, called the numerical method, is represented by the numerical Riemann integral and is given by the following formula:

$$Res(a) = \frac{1}{\omega+1} \sum_{n=0}^{\omega} g[t_n]\Delta s, \tag{24}$$

The total interval $[b,c]$ is divided into v equal subintervals, each of length Δs, where Δs, represents the uniform spacing between evaluation points. In this method, the function $g(t)$ is defined as $g[t] = [\ell(u_k(t)) + \mathcal{N}(u_k(t)) - f(t)]^2$. Here, v denotes the number of subintervals used in the approximation process. Nonetheless, applying analytical methods for complex nonlinear equations is frequently difficult. In contrast, a numerical method grounded in Riemann integration offers a more effective alternative, as it accelerates computations while preserving a high degree of accuracy, making it very valuable in advanced computational fields.

4. Convergence Analysis and Approximate Solutions with Numerical Results

4.1. Convergence Analysis of the ADM and the OADM

This subsection studies the convergence of ADM and OADM in relation to **Equation 9**. The fundamental prerequisites for achieving convergence are established, together with the assessment of the mistake produced by this method. The fundamental findings are delineated within the following theories[2,31]. To explain it, initially the following steps are selected.

$$u_0 = u_0(t),$$
$$u_1 = B[u_0],$$
$$u_2 = B[u_0 + u_1],$$
$$\vdots$$
$$u_{k+1} = B[u_0 + u_1 + \cdots + u_k], \quad \textbf{(25)}$$

Consider $u_0(t)$ to denote the approximate solution derived from the initial iteration, and the operator B is represented by the subsequent relation:

$$B[u_p] = S_p - \sum_{k=0}^{P} u_k, \; p = 0,1,2, \ldots \quad \textbf{(26)}$$

such that S_p signifies the solution obtained from the proposed methods. The solution can be expressed using **Equations 25** and **26** as follows:

$$u(t) = \sum_{k=0}^{\infty} u_k(t), \quad \textbf{(27)}$$

4.1.1 Theorem

suppose B, as defined in **Equation 26**, is the operator assignment from the Hilbert space H to H. The series $u(t) = \sum_{k=0}^{\infty} u_k(t)$ is converges**,** if $\exists\, 0 < \delta < 1$ such that $\|B[u_0 + u_1 + \cdots + u_{k+1}]\| \le \delta \,\|B[u_0 + u_1 + \cdots + u_k]\|$, (Specifically $\|u_{k+1}\| \le \delta \|u_k\|, \forall\, k \in \mathbb{N} \cup \{0\}$).

Proof: see[31].

4.1.2 Theorem

if the series solution $u(t) = \sum_{k=0}^{\infty} u_k(t)$, as presented in **Equation 27**, converges, it constitutes the exact solution of the nonlinear **Equation 9**.

Proof: see[31].

4.1.3 Theorem

let the series solution $\sum_{k=0}^{\infty} u_k(t)$ defined in **Equation 27** be approximated by the truncated series $u(t) = \sum_{k=0}^{j} u_k(t)$, consider the solution $u(t)$ of **Equation 9**. The maximum mistake $E_j(t)$ can be derived from the following formula[2,31]:

$$E_j(t) \le \frac{\delta^{j+1}}{1-\delta} \|u_0\|, \quad \textbf{(28)}$$

To summarize. 4.1.1 and 4.1.2 theorems illustrate that the nonlinear solution obtained from the ADM for **Equation 9**, using the iteration formulas defined in **Equation 25**, converges to the exact solution provided that there exists a $0 < \delta < 1$ such that $\|u_{k+1}\| \le \delta \|u_k\|$. In other terms, if the parameter values are specified for each $k \in \mathbb{N} \cup \{0\}$, then:

$$\beta_i = \begin{cases} \frac{\|u_{k+1}\|}{\|u_k\|}, & \|u_k\| \ne 0 \\ 0, & \|u_k\| = 0 \end{cases} \quad \textbf{(29)}$$

The series solution $\sum_{k=0}^{\infty} u_k(t)$ for **Equation 9** converges to an exact solution $u(t)$ when $0 \le \beta_i < 1$ for every $i \in \mathbb{N} \cup \{0\}$. The maximum error of the truncated absolute, based on Theorem 4.3, is determined to be $\left\|u(t) - \sum_{k=0}^{j} u_k(t)\right\| \le \frac{\beta^{j+1}}{1-\beta} \|u_0\|$, where $\beta = max\{\beta_i, i = 0,1,2, \ldots, j\}$.

4.2. Approximate Solutions and Numerical Results

Based on the convergence results specified in the previous subsection, approximate solutions and numerical results for the proposed applications are now presented.

4.2.1. Approximate Solution and Numerical Results for the DBFME

Now, we will study DBFME in the case $M = s = F = 1$.

$$u''(t) - u(t) - u^2(t) + 1 = 0, \quad u'(0) = 0, u(1) = 0, \quad \textbf{(30)}$$

By applying the ADM,

$u''(t) = u^2(t) + u(t) - 1,$

$\ell u = u^2(t) + u(t) - 1, \;\; u'(0) = 0, u(1) = 0,$ **(31)**

where ℓ is a second-order differential operator, $\ell(\cdot) = \frac{d^2u}{dt^2}(\cdot)$.

It is clear that ℓ^{-1} is invertible and is given by $\ell^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) drdr$.

Operating ℓ^{-1} on both sides of **Equation 31** and using the boundary conditions, we get:

$\ell^{-1}(\ell u) = \ell^{-1}(u^2(t) + u(t) - 1),$

$u(t) - u(0) - t\,u'(0) = \int_0^t \int_0^t (u^2(r) + u(r) - 1) drdr,$

The boundary conditions and the Maclaurin series yielded two approximate initial values: -3.30278 and 0.302776, indicating the existence of double solutions [32]. To determine the optimal value, the MER_k was calculated for each of them, resulting in 0.000893735 for the negative value and 4.64344×10^{-9} for the positive value. This study adopted the positive value of its effectiveness in decreasing error.

$u(t) = 0.302776 - \frac{t^2}{2} + \int_0^t \int_0^t \left(u^2(r) + u(r)\right) drdr \;\; =$ **(32)**

Suppose the unknown function $u(t)$ is represented as an infinite series: $u(t) = \sum_{n=0}^{\infty} u_n(t)$.

Consequently, the initial Adomian polynomial is expressed as: $A_0 = \mathcal{N}\left(u_0(t)\right) = u_0^2(t) + u_0(t)$.

The alternative formula given in **Equation 17** was used to calculate A_k. The nonlinear boundary is defined by the following equation:

$\mathcal{N}(u) = (u_0 + u_1 + u_2 + u_3 + \cdots)^2 + (u_0 + u_1 + u_2 + u_3 + \cdots)$ **(33)**

Based on **Equations 32** and **33**, the following iterations were obtained:

$u_0(t) = 0.302776 - \frac{t^2}{2},$

$u_1(t) = 0.302776 \; - 0.302775 \; t^2 - \cdots,$

$u_2(t) = 0.302776 \; - 0.302775 \; t^2 - 0.066898 \; t^4 + \cdots,$

$u_3(t) = 0.302776 \; - 0.302775 \; t^2 - 0.066898 \; t^4 + 0.00833333 \; t^6 + \cdots,$

$\vdots$

Utilizing the OADM,

$u_0(t) = 0.302776 - \frac{t^2}{2} - at,$

$u_1(t) = at + \ell^{-1}\left(0.302776 - \frac{t^2}{2}\right),$

$u_{k+1}(t) = \ell^{-1}[A_k], \; k \geq 1,$ and $\mathcal{N}(u) = \sum_{k=0}^{\infty} A_k$ **(34)**

Based on **Equations 33** and **34**, the following iterations were obtained:

$u_2(t) = 0.302776 - \frac{t^2}{2} + t^2 \; (0.197225 \; - \cdots,$

$u_3(t) = 0.302776 - \frac{t^2}{2} + t^2 \; (0.197225 \; - 0.267592 \; at - \cdots,$

$u_4(t) = 0.302776 - \frac{t^2}{2} + t^2 \; (0.197225 \; - 0.267592 \; at - 0.066898 \; t^2 + \cdots,$

$\vdots$

Due to the unavailability of the exact solution, the maximum error remainder (MER_k) was used to evaluate the accuracy of approximate solutions; it is defined as follows[2]:

$MER_k = max_{b \leq t \leq c} \left| u''(t) - s^2\, u(t) - F\, s\, u^2(t) + \frac{1}{M} \right|,$ **(35)**

Table 1 illustrates the considerable disparities in the maximum error remainder values between the ADM and the OADM, indicating the OADM's superiority in yielding satisfactory outcomes.

Table 1. Maximum error remainder of the ADM and the OADM for **Equation 30**, when $k = 7$.

Interval	$\mathbf{MER_k}$ of ADM	$\boldsymbol{a}$	$\mathbf{MER_k}$ of OADM
[0,1]	$1.492948264909159 \times 10^{-6}$	-0.05846823847	$4.64343877194448 \times 10^{-9}$
[0,2]	0.0005776516369266904	-0.10929936755	$8.140532778977416 \times 10^{-7}$
[0,3]	0.49570237616199364	-0.26331985063	0.02587416719922242

Figure 1 shows the logarithmic plots for the MER_k values obtained by the ADM and OADM. The graph clearly shows that MER_k decreases more quickly with OADM than with ADM, emphasizing that OADM is better at achieving faster results and greater accuracy in the same number of iterations.

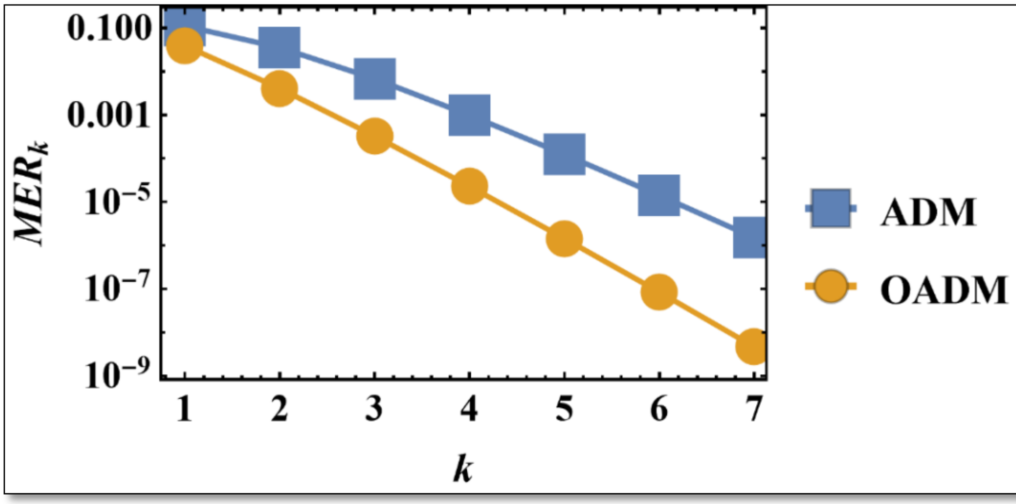

Figure 1. Logarithmic plots for the MER_k when $M = s = F = 1$ for the DBFME by using the proposed methods.

Table 2 indicates that all computed values of the ADM and OADM, which confirming their convergence according to the defined convergence criterion in section 4.

Table 2. The value of β_i to the approximate solutions of the proposed methods for $k = 1$ to 7 when $M = s = F = 1$ for the DBFME.

$\boldsymbol{\beta_i}$	ADM	OADM
$\boldsymbol{\beta_1}$	0.03323500502987307	0.0004975049182286283
$\boldsymbol{\beta_2}$	0.08492618802306826	0.030559941730887562
$\boldsymbol{\beta_3}$	0.08568848874114562	0.01352610112440358
$\boldsymbol{\beta_4}$	0.08272010589508128	0.04206038853545265
$\boldsymbol{\beta_5}$	0.08102859743768202	0.02328385201951671
$\boldsymbol{\beta_6}$	0.0798360456841542	0.03449729129252715

The convergence rate when $M = s = F = 1$ for the DBFME by using the ADM and OADM was calculated using the following logarithmic formula:

$$\vartheta = \log\left[\frac{MER_7}{MER_6}\right] / \log\left[\frac{MER_6}{MER_5}\right], \quad \textbf{(36)}$$

The results indicate that the convergence rate for both methods is approximately one, implying that both ADM and OADM show linear convergence. i.e. $\vartheta \simeq 1$.

Additional cases of the $F, s,$ and M parameters will be studied at $k = 7$.

Case 1: The values F and s are both fixed at 1, while the selected values of M are $2, 3.5$, and 7.

Table 3 demonstrates that the OADM gives better results in comparison to the ADM in all studied hypotheses. Increasing the value of M is associated with a continuous decrease in the MER_k, signifying an improvement in the accuracy and stability of the OADM. This signifies its superiority and efficacy in solving the DBFME approximately.

Table 3. The comparison between the MER_7 when $F = s = 1$, and versus the value of M for the DBFME.

$F = s = 1$, $M = 2$.			$F = s = 1$, $M = 3.5$.		$F = s = 1$, $M = 7$.	
Interval	MER_7 of ADM	MER_7 of OADM	MER_7 of ADM	MER_7 of OADM	MER_7 of ADM	MER_7 of OADM
[0,1]	9.661239×10^{-8}	4.369446×10^{-9}	1.303908×10^{-8}	5.940003×10^{-11}	1.383677×10^{-9}	$8.8641498 \times 10^{-12}$
[0,2]	0.0001187945	1.500569×10^{-7}	0.0000301383	5.840067×10^{-7}	5.542708×10^{-6}	3.8101008×10^{-7}
[0,3]	0.0005495618	9.134470×10^{-6}	0.0000952675	1.189426×10^{-6}	0.00004095639	4.0543435×10^{-6}
[0,4]	0.3202575444	0.00584496735	0.0296527759	0.00308419549	0.00849940315	0.000192704505

Figures 2a and **2b** and **Table 3** illustrate that increasing the parameter M helps increase result accuracy and reduces error values, highlighting parameter M's impact on the accuracy of the solutions. The OADM, illustrated in **Figure 2b**, had results that were more accurate than the ADM results shown in **Figure 2a**.

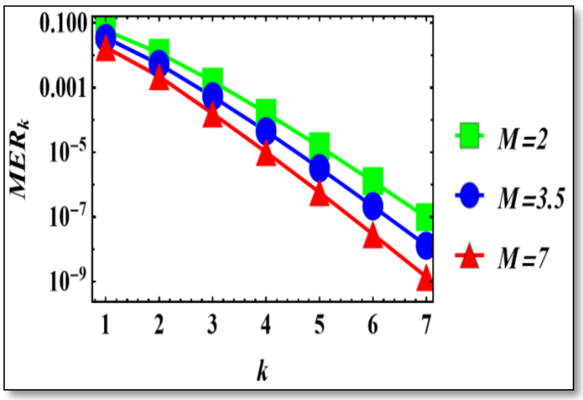

Figure 2(a). plots for the approximate solution the MER_k obtained by the ADM for the case 1 on [0,1].

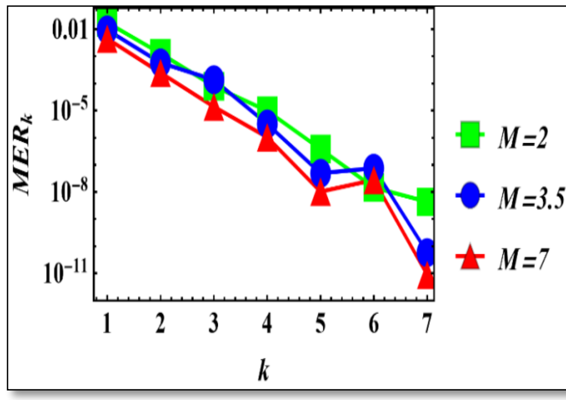

Figure 2(b). plots for the approximate solution the MER_k obtained by the OADM for the case 1 on [0,1].

Case 2: The values M and F are both fixed at 1, while the selected values of s are $0.1, 0.5$ and 0.8.

Table 4 shows the values of MER_k for the ADM and OADM of approximation solutions obtained from DBFME, using different values of a parameter of s. The results demonstrate a distinct superiority for the OADM compared to the ADM across all selected intervals, since the OADM generated many fewer errors than the ADM, indicating its high accuracy. These findings confirm the OADM as a precise and dependable instrument for solving nonlinear equations.

Table 4. The comparison between the MER_7 when $M = F = 1$, and versus the value of s for the DBFME.

M = **F** = 1, $s = 0.1$.			**M** = **F** = 1, $s = 0.5$.		**M** = **F** = 1, $s = 0.8$.	
Interval	MER_7 of ADM	MER_7 of OADM	MER_7 of ADM	MER_7 of OADM	MER_7 of ADM	MER_7 of OADM
[0,1]	1.474766×10^{-15}	7.155734×10^{-18}	2.7679940×10^{-9}	$5.1373558 \times 10^{-12}$	2.015801×10^{-7}	3.821499×10^{-10}
[0,2]	1.147415×10^{-12}	5.148659×10^{-14}	6.9089895×10^{-7}	5.8487549×10^{-8}	0.000074264383	1.294895×10^{-7}
[0,3]	1.024565×10^{-7}	8.171945×10^{-10}	0.0199945383	0.000901201998	0.289964697209	0.01447301795

Figures 3a and **3b** and **Table 4** illustrate that an increase in the parameter value s leads to increased error rates. **Figure 3b** demonstrates that the OADM provides better accuracy than the ADM shown in **Figure 3a**.

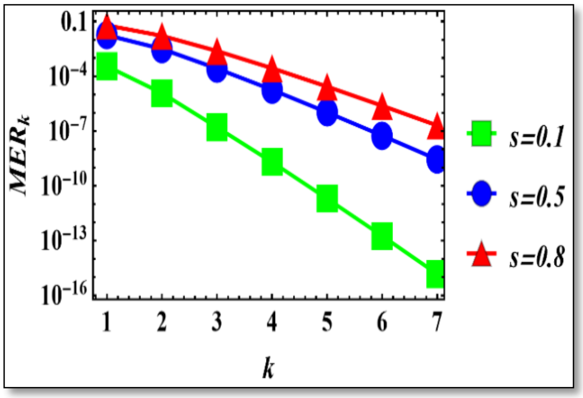

Figure 3(a). Logarithmic plots for the MER_k for the case 2 by applying the ADM on [0,1].

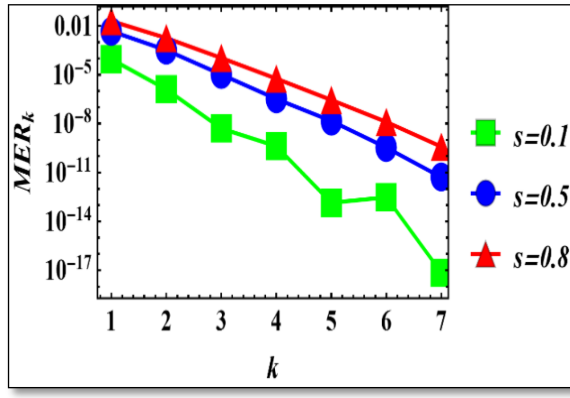

Figure 3(b). Logarithmic plots for the MER_k for the case 2 by applying the OADM on [0,1].

Case 3: The values M and s are both fixed at 1, while the selected values of F are 0, 0.5 and 1.5. **Table 5** demonstrates the superiority of the OADM over the ADM in all evaluated instances, as the OADM produced the fewest MER_k values. Despite the rise in values, the OADM delivered precise findings. In the situation of $F = 1.5$ and the interval [0,3], the ADM exhibited escalating mistakes, whereas the OADM preserved its accuracy.

Table 5. The comparison between the MER_7 when $M = s = 1$, and versus the value of F for the DBFME.

M = **s** = 1, **F** = **0.**			**M** = **s** = 1, **F** = **0.5.**		**M** = **s** = 1, **F** = **1.5.**	
Interval	MER_7 of ADM	MER_7 of OADM	MER_7 of ADM	MER_7 of OADM	MER_7 of ADM	MER_7 of OADM
[0,1]	3.775787×10^{-12}	1.312321×10^{-13}	1.932277×10^{-7}	7.351319×10^{-10}	5.597033×10^{-6}	1.604293×10^{-8}
[0,2]	5.951329×10^{-8}	1.976703×10^{-9}	0.000237594505	6.078670×10^{-7}	0.000857085999	0.00003027982
[0,3]	0.000016230	4.907516×10^{-7}	0.001099141727	0.0000177224524	43.24703142686	0.74124493288

Figures 4a and **4b** and **Table 5** indicate an increase in the parameter value F negatively affects result accuracy, as it leads to increased error rates. **Figure 4b** demonstrates that the OADM attains more accuracy after seven iterations compared to the ADM depicted in **Figure 4a**.

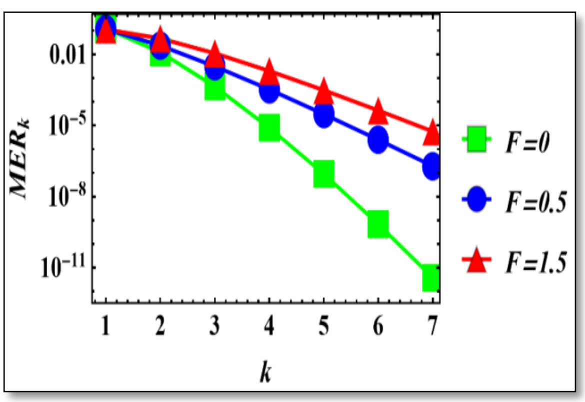

Figure 4(a). Logarithmic plots for the MER_k for the case 3 by using the ADM on [0,1].

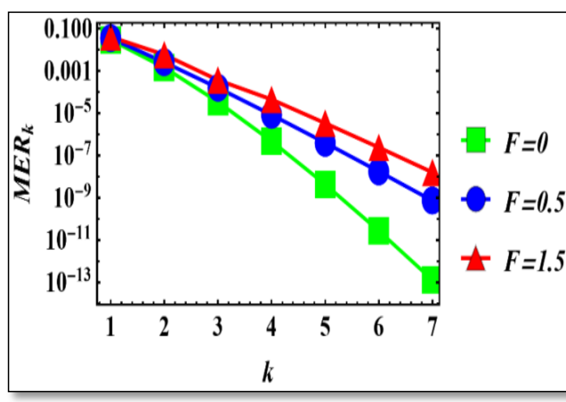

Figure 4(b). Logarithmic plots for the MER_k for the case 3 by using the OADM on [0,1].

Table 6 will examine the results from the ADM and OADM for the DBFME and compare them with the operational matrix methods, specifically the Bernoulli operational matrix method (BrOM), the Bernstein operational matrix method (BOM), and the shifting Legendre operational matrix method (LOM), as explained in[2]. Here, P represents the parameter under study (i.e., M, s, or F), whereas V indicates the chosen value of that parameter. **Table 6** shows the numerical results obtained using these methods. The OADM demonstrates superiority over other methods, evidenced by the lowest values of MER_k across all cases, indicating its exceptional accuracy compared to other methods. The MER_k values diminish as M increases, whereas they escalate with rising s and F; this signifies the impact of these parameters on the solution's accuracy.

Table 6. The comparison between the MER_9 for the DBFME by proposed methods and methods in[2].

P	V	$\mathbf{MER_9}$ of BrOM[2]	$\mathbf{MER_9}$ of BOM[2]	$\mathbf{MER_9}$ of LOM[2]	$\mathbf{MER_9}$ of ADM	$\mathbf{MER_9}$ of OADM
M	2	$2.8034747706456 \times 10^{-8}$	$2.8030427701164 \times 10^{-8}$	$2.80347478547743 \times 10^{-8}$	$5.04795989 \times 10^{-10}$	2.631586×10^{-12}
M	3	$1.4228995742077 \times 10^{-8}$	$1.4228955272713 \times 10^{-8}$	$1.42289990016226 \times 10^{-8}$	$8.29334942 \times 10^{-11}$	1.073622×10^{-13}
M	4	$1.0910099730761 \times 10^{-8}$	$1.0910070058316 \times 10^{-8}$	$1.09101025652994 \times 10^{-8}$	$2.47230667 \times 10^{-11}$	3.586190×10^{-14}
s	0.5	$3.0763325700473 \times 10^{-7}$	$3.0764346257983 \times 10^{-7}$	$3.0763325903870 \times 10^{-7}$	$5.86436889 \times 10^{-12}$	3.058317×10^{-15}
s	2	0.0000047476304617	0.0000047476286068	0.0000047476304628	0.0000270814030	5.3010699×10^{-8}
s	3	0.0000153714428449	0.0000153714398763	0.0000153714428436	0.0021172079017	7.5416825×10^{-6}
F	0	$2.572762360107 \times 10^{-9}$	$2.5678844606247 \times 10^{-9}$	$2.5727621084709 \times 10^{-9}$	$5.16529915 \times 10^{-17}$	1.397212×10^{-18}
F	2	0.0000030354047633	0.0000030353988951	0.0000030354047617	$2.952764015 \times 10^{-7}$	2.094329×10^{-10}
F	4	0.0000123160965391	0.0000123160923355	0.0000123160965380	$7.236605329 \times 10^{-6}$	5.0263694×10^{-9}

4.2.2. Approximate Solution and Numerical Results of the Blasius Equation

This subsection discusses the efficiency of the proposed methods when applied to the Blasius equation in obtaining approximate solutions.

$$u'''(t) + 0.5\, u''(t)\, u(t) = 0,\ \ u(0) = u'(0) = 0, u''(0) = 0.3320573 \quad \textbf{(37)}$$

Applying the ADM,

$$u'''(t) = -0.5\, u''(t)\, u(t) \quad \textbf{(38)}$$

$$\ell u = -0.5\ \ u''(t)\, u(t),\ \ u(0) = u'(0) = 0, u''(0) = 0.3320573, \quad \textbf{(39)}$$

where ℓ is a third-order differential operator, $\ell(\cdot) = \frac{d^3u}{dt^3}(\cdot)$.

It is clear that ℓ^{-1} is invertible and is given by $\ell^{-1}(\cdot) = \int_0^t \int_0^t \int_0^t (\cdot) drdrdr$,

Operating ℓ^{-1} on both sides of **Equation 39** and using the initial conditions gives,

$$\ell^{-1}(\ell u) = -\ell^{-1}\big(0.5\, u''(t)\, u(t)\big),$$

$$u(t) - u(0) - t\, u'(0) - \frac{t^2}{2}\, u''(0) = -0.5\ \int_0^t \int_0^t \int_0^t \big(u''(r) u(r)\big) drdr\, dr,$$

$$u(t) = 0.16602865\, t^2 - 0.5\ \int_0^t \int_0^t \int_0^t \big(u''(r)\, u(r)\big) drdr\, dr,$$

$$u_{k+1}(t) = 0.16602865\, t^2 - 0.5\ \int_0^t \int_0^t \int_0^t \big(u_k''(r)\, u_k(r)\big) drdr\, dr,$$

$$u_{k+1}(t) = 0.16602865\, t^2 - 0.5\, \ell^{-1}(A_k), \text{ for } k \geq 0. \quad \textbf{(40)}$$

Suppose the unknown function $u(t)$ is represented as an infinite series: $u(t) = \sum_{k=0}^{\infty} u_k(t)$.

Consequently, the initial Adomian polynomial is expressed as: $A_0 = \mathcal{N}\big(u_0(t)\big) = u_0''(t)\, u_0(t)$.

The Adomian polynomials were calculated based on the following formula:

$$\mathcal{N}(u) = \big[(u_{0_{tt}} + u_{1_{tt}} + u_{2_{tt}} + u_{3_{tt}} + \cdots)(u_0 + u_1 + u_2 + u_3 + \cdots)\big]. \quad \textbf{(41)}$$

The following iterations were obtained using **Equations 40** and **41**:

$u_0(t) = 0.16602865\, t^2$,

$u_1(t) = 0.166029\, t^2 - 0.000459425\ \ t^5$,

$u_2(t) = 0.166029\, t^2 - 0.000459425\ \ t^5 + 2.49719 \times 10^{-6}\, t^8$,

$u_3(t) = 0.166029\, t^2 - 0.000459425\ \ t^5 + 2.49719 \times 10^{-6}\, t^8 - 1.4277 \times 10^{-8}\ \ t^{11}$,

⋮

Utilizing the OADM,

$u_0(t) = 0.16602865\, t^2 - at,$

$u_1(t) = at + \ell^{-1}(0.16602865\, t^2),$

$u_{k+1}(t) = -0.5\, \ell^{-1}[A_k], k \geq 1, \quad \text{and } \mathcal{N}(u) = \sum_{k=0}^{\infty} A_k\, .$ **(42)**

The following iterations were obtained using **Equations 41** and **42**:

$u_2(t) = 0.166029\, t^2 - 0.000459425\, t^5 + 0.000345893\, a^2\, t^6 - \cdots,$

$u_3(t) = 0.166029\, t^2 - 0.000459425\, t^5 - 0.000345893\, a^2\, t^6 + 2.49719 \times 10^{-6}\, t^8 + \cdots,$

$u_4(t) = 0.166029\, t^2 - 0.000459425\, t^5 + 2.49719 \times 10^{-6}\, t^8 - 0.0000308833\, a^3\, t^8 + \cdots,$

Table 7 shows that the OADM clearly outperforms the ADM over all intervals. The ADM failure becomes clear in the fifth interval due to an increase in error; however, the OADM provided a satisfying outcome over the same interval, highlighting the importance of the optimal parameter in enhancing results.

Table 7. The comparison between the MER_7 of the proposed methods for the Blasius Equation.

Interval	$\boldsymbol{MER_7}$ of ADM	$\boldsymbol{a}$	$\boldsymbol{MER_7}$ of OADM
[0,1]	$1.137978600240785 \times 10^{-15}$	0.000199281350018	$4.85722573273506 \times 10^{-17}$
[0,2]	$9.344835572022703 \times 10^{-9}$	0.003125342126829	$2.70498526355745 \times 10^{-10}$
[0,3]	0.00009771091907317953	0.014858737889814	$1.834691472815675 \times 10^{-6}$
[0,4]	0.06498143170294823	0.040799939122759	0.0010876971553042125
[0,5]	9.422407028848566	0.134141540286137	0.13843824779718616

Figure 5 illustrates that the MER_k noticeably decreases with an increase in iterations for both the ADM and OADM, signifying the convergence of the solutions. However, the accuracy and convergence speed for the OADM are the highest.

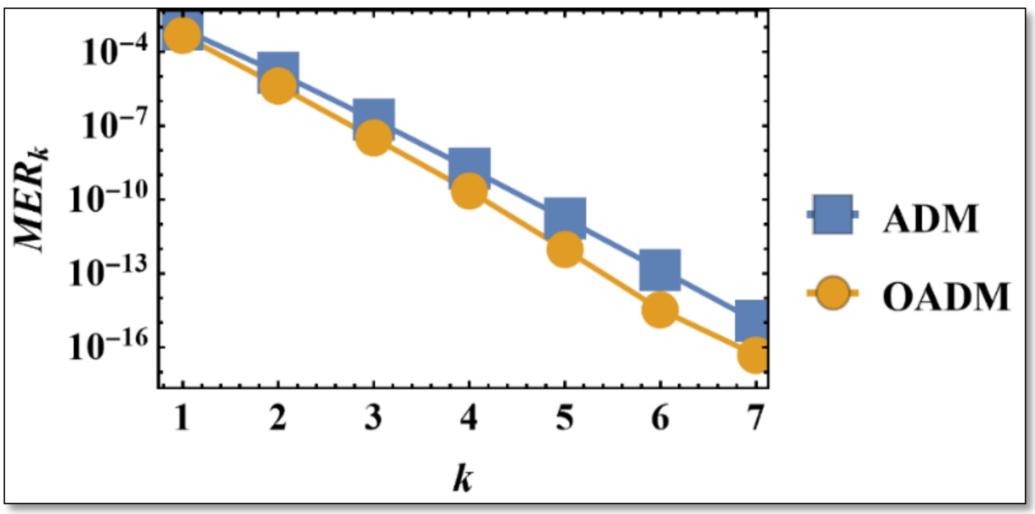

Figure 5. Logarithmic plots for the MER_k versus k from 1 to 7 for the Blasius Equation on [0,1].

Table 8 demonstrates that all β_i values are less than 1. This confirms that they are convergent. Furthermore, results demonstrate the superiority of OADM compared to ADM regarding convergence speed.

Table 8. The value of β_i to the approximate solutions of the proposed methods for $k = 1$ to 7 for the Blasius Equation.

β_i	ADM	OADM
β_1	0.000010445788497210762	0.000001180933040111432
β_2	0.005450761440848835	0.0011464518096362094
β_3	0.005520943815982292	0.004300797556663325
β_4	0.005498991627724247	0.001079203468290825
β_5	0.005470076398964747	0.003225241473644877
β_6	0.005450210695970095	0.002157989676305241

The rates of convergence for **Equation 37** were computed utilizing the ADM and OADM, in accordance with **Equation 36**, for values of k (from 1 to 7). The results indicated that both methods show linear convergence because the convergence rates are approximately one.

This subsection examines how effective the proposed methods are in obtaining approximate solutions to the Blasius equation, using both the ADM and OADM, with results previously documented in[3]. **Table 9** demonstrates the efficacy of the OADM in minimizing the MER_k achieving $4.85722573274 \times 10^{-17}$. **Figure 6** illustrates a rapid and continuous decrease in error corresponding to an increase in the number of iterations. This result confirms the effectiveness of the OADM in achieving accuracy and convergence.

Table 9. The comparison between the MER_k versus k from 1 to 7 for the Blasius equation on [0,1].

k	MER_k of BrOM[3]	MER_k of BOM[3]	MER_k of LOM[3]	MER_k of ADM	MER_k of OADM
3	0.01481939170059307	0.014819391700592988	0.0148193917005929	$1.7574041905 \times 10^{-7}$	$3.1939353065 \times 10^{-8}$
4	0.01142701573855896	0.011427015738563973	0.0114270157385584	$1.8275754729 \times 10^{-9}$	$2.272030654 \times 10^{-10}$
5	0.00041970597546396	0.000419705974763596	0.0004197059754625	$1.6866567171 \times 10^{-11}$	$9.389954092 \times 10^{-13}$
6	0.00017127579189292	0.000171275791933212	0.0001712757919438	$1.4328468967 \times 10^{-13}$	$3.337607968 \times 10^{-15}$
7	0.00002513798889495	0.000025137988745927	0.0000251379883112	$1.1379786002 \times 10^{-15}$	$4.857225733 \times 10^{-17}$

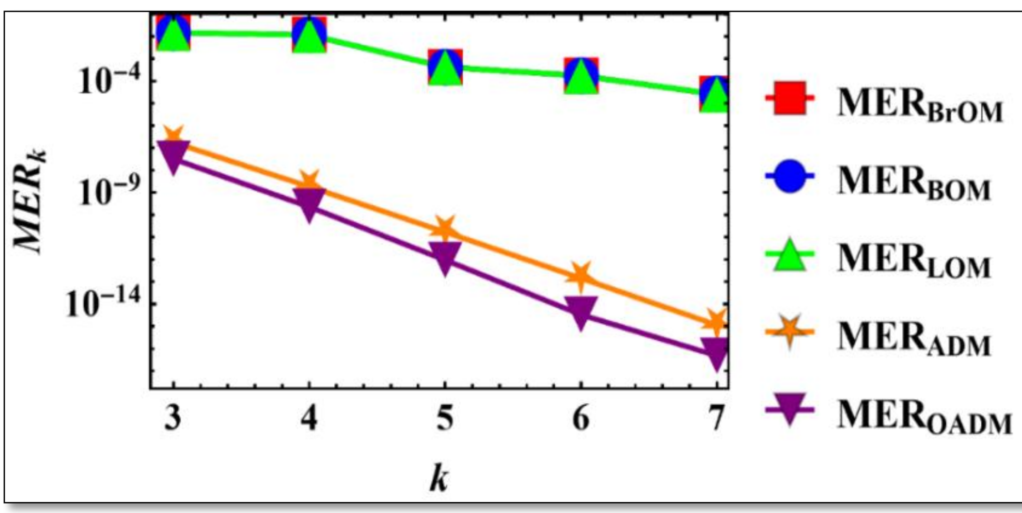

Figure 6. Logarithmic plots for the MER_k for the Blasius Equation by using the proposed methods on [0,1].

4.2.3. Approximate Solution and Numerical Results of the Falkner-Skan Equation

This subsection discusses the accuracy of the ADM and OADM when they are applied to the Falkner-Skan equation to get approximate solutions at $\theta = 0.5$ and $\varepsilon = 0.1$.

$u'''(t) + u''(t)\, u(t) + 0.5\left[(0.1)^2 - \left(u'(t)\right)^2\right] = 0, u(0) = 0, u'(0) = 0.9, u''(0) = -0.832666$ **(43)**

By using the ADM,

$u'''(t) = -0.005 - u''(t)\, u(t) + 0.5\left(u'(t)\right)^2, u(0) = 0, u'(0) = 0.9, u''(0) = -0.832666.$

$\ell u = -0.005 - u''(t)\, u(t) + 0.5\left(u'(t)\right)^2,$ **(44)**

where ℓ is a third-order differential operator, $\ell u = \frac{d^3u}{dt^3}$.

It is clear that ℓ^{-1} is invertible and is given by $\ell^{-1}(\cdot) = \int_0^t \int_0^t \int_0^t (\cdot) dr dr dr$,

Operating ℓ^{-1} on both sides of **Equation 44** and using the initial conditions gives

$\ell^{-1}(\ell u) = \ell^{-1}\left(-0.005 - u''(t)\, u(t) + 0.5\left(u'(t)\right)^2\right)$, then we get:

$u(t) - u(0) - t\, u'(0) - \frac{t^2}{2} u''(0) = \int_0^t \int_0^t \int_0^t \left(-0.005 - u''(r)\, u(r) + 0.5\left(u'(r)\right)^2\right) dr dr\, dr,$

$u_{k+1}(t) = 0.9\, t - 0.416333\, t^2 - (0.005/6)\, t^3 + \int_0^t \int_0^t \int_0^t \left(u_k''(r)\, u_k(r) + 0.5\left(u_k'(r)\right)^2\right) dr dr\, dr,$ for $k \geq 0$ **(45)**

Let us assume that the unknown function $u(t)$ may be expressed as an infinite series: $u(t) = \sum_{k=0}^{\infty} u_k(t)$. The initial Adomian polynomial is expressed as follows:

$A_0 = \mathcal{N}\left(u_0(t)\right) = -u_0''(t)\, u_0(t) + 0.5\left(u_0'(t)\right)^2.$

The Adomian polynomials were calculated based on the following formula:

$\mathcal{N}(u) = \left[-\left(u_{0_{tt}} + u_{1_{tt}} + u_{2_{tt}} + u_{3_{tt}} + \cdots\right)\left(u_0 + u_1 + u_2 + u_3 + \cdots\right) + 0.5\left(u_{0_t} + u_{1_t} + u_{2_t} + u_{3_t} + \cdots\right)^2\right].$ **(46)**

Based on **Equations 45** and **46**, the following iterations were obtained:

$u_0(t) = 0.9\, t - 0.416333\, t^2 - (0.005/6)\, t^3,$

$u_1(t) = 0.9\, t - 0.416333\, t^2 + 0.0666667\, t^3 + 0.0000375\, t^5 - \cdots,$

$u_2(t) = 0.9\, t - 0.416333\, t^2 + 0.0666667\, t^3 - 0.003\, t^5 + 0.000462592\, t^6 - \cdots,$

$u_3(t) = 0.9\, t - 0.416333\, t^2 + 0.0666667\, t^3 - 0.003\, t^5 + 0.000462592\, t^6 + 0.000161111\, t^7 - \cdots,$

$\vdots$

By applying the OADM,

$u_0(t) = (0.9\, t - 0.416333\, t^2 - (0.005/6)\, t^3) - at,$

$u_1(t) = at + \ell^{-1}(0.9\, t - 0.416333\, t^2 - (0.005/6)\, t^3),$

$u_{k+1}(t) = \ell^{-1}[A_k],\ k \geq 1,$ and $\mathcal{N}(u) = \sum_{k=0}^{\infty} A_k$ **(47)**

Based on **Equations 46** and **47**, the following iterations were obtained:

$u_2(t) = 0.9t - 0.416333t^2 - 0.000833333t^3 + 1/6\,(0.9a - \cdots,$

$u_3(t) = 0.9t - 0.416333t^2 - 0.000833333t^3 + 0.0833333\, a^2t^3 + 1/6\,(0.9\, a - \cdots,$

$u_4(t) = 0.9t - 0.416333t^2 - 0.000833333t^3 + 0.0833333\, a^2t^3 + 1/6\,(0.9\, a - a^2)t^3 + \cdots,$

Table 10 indicates the superiority of the OADM over the ADM in terms of accuracy, as the OADM maintained low error values over different intervals, while the ADM errors increased significantly, especially at the interval [0,3].

Table 10. The comparison for the MER_7 when $\theta = 0.5$ and $\varepsilon = 0.1$ for the Falkner-Skan Equation by using the proposed methods.

Interval	$\mathbf{MER_7}$ of ADM	$\boldsymbol{a}$	$\mathbf{MER_7}$ of OADM
[0,1]	$1.088765707590866 \times 10^{-7}$	0.01476666572	$8.265401807427963 \times 10^{-9}$
[0,2]	0.00030938362349317017	0.01190500549	0.000006954338474787769
[0,3]	0.11134611991562338	0.08962893340	0.01786768847740916

Figure 7 illustrates that the MER_k value decreases significantly with an increase in the number of iterations for both the ADM and OADM, indicating the convergence of solutions. However,

the results demonstrate the superiority of the OADM for accuracy and convergence speed in comparison to the ADM.

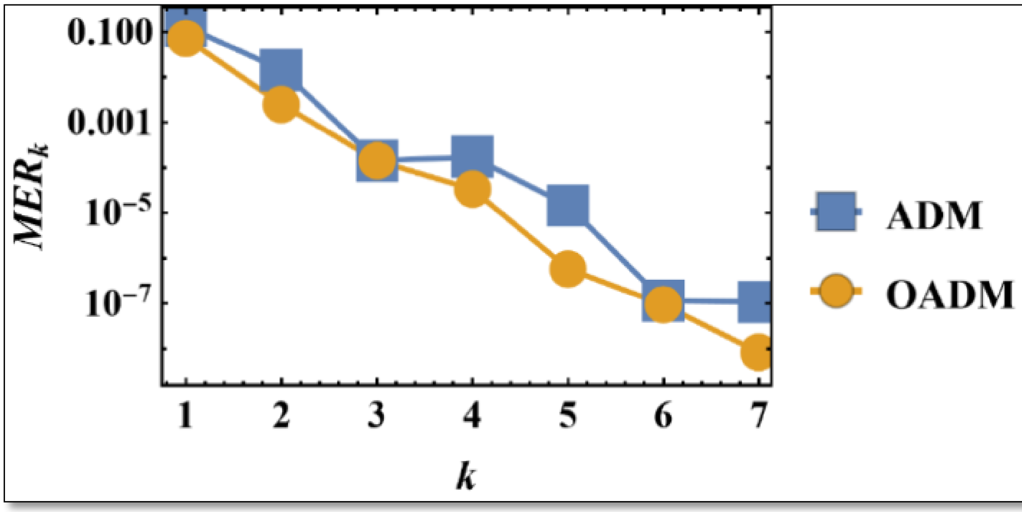

Figure 7. Logarithmic plots for the MER_k versus k from 1 to 7, for **Equation 43** on [0,1].

Table 11 shows the convergence of approximate solutions towards the exact solution, and also note the superiority of OADM over ADM in the speed of convergence.

Table 11. The value of β_i to the approximate solutions when $\theta = 0.5$ and $\varepsilon = 0.1$ for the Falkner-Skan equation by using proposed methods

β_i	ADM	OADM
β_1	0.0026005419225494635	0.0000027060516326456 62
β_2	0.03461584661944827	0.0036646920293140813
β_3	0.008376981911702667	0.010445942798224172
β_4	0.20482807365297173	0.013079793923822199
β_5	0.06306081710416353	0.004676550287729715
β_6	0.008326745655578864	0.004124425763910496

In addition, this subsection compared the results of iterative methods with the operational matrices methods, as presented in[1], for the approximate solution of the Falkner-Skan equation. **Table 12** illustrates a comparison between BrOM, BOM, LOM, ADM, and OADM for the MER_k. The analysis demonstrates that the OADM significantly reduces the error, hence affirming its efficacy and precision in nearly solving the Falkner-Skan equation. **Figure 8** shows that as the number of iterations increases, the error quickly and persistently decreases, proving that the OADM is effective in reaching high accuracy and better results compared to other methods.

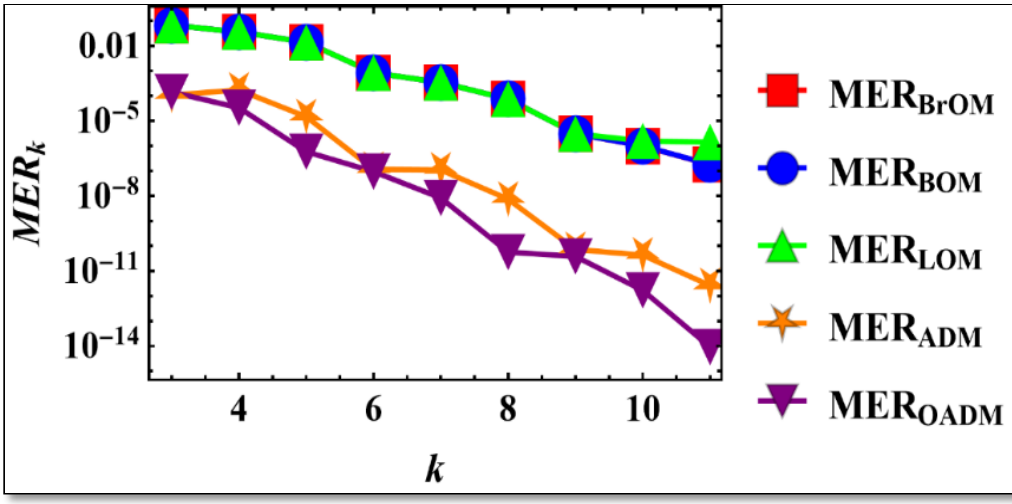

Figure 8. The comparison of the solutions, when $\theta = 0.5$ and $\varepsilon = 0.1$ for **Equation 43** on $[0,1]$.

Table 12. The comparison for the MER_k between the proposed methods, when $\theta = 0.5$ and $\varepsilon = 0.1$ for **Equation (43)** on $[0,1]$.

k	$\mathbf{MER_k}$ of BrOM[1]	$\mathbf{MER_k}$ of BOM[1]	$\mathbf{MER_k}$ of LOM[1]	$\mathbf{MER_k}$ of ADM	$\mathbf{MER_k}$ of OADM
3	0.06638210579917353	0.06638210579917359	0.06638210579917331	0.0001073093	0.00014187095
4	0.03585217953692066	0.03585217953692854	0.03585217953690067	0.0001713611	0.00003331139
5	0.01379314996965419	0.01379314991950075	0.01379314996965491	0.0000146344	$5.836757671 \times 10^{-7}$
6	0.00083723186488526	0.00083723186485906	0.00083723186488610	1.155865×10^{-7}	$9.356359898 \times 10^{-8}$
7	0.00034011248679871	0.00034011249043919	0.00034011248679977	1.088766×10^{-7}	$8.265401807 \times 10^{-9}$
8	0.00007441899401172	0.00007441897180771	0.00007441899400839	7.400823×10^{-9}	$5.638101097 \times 10^{-11}$
9	0.00000307279760042	0.00000307277674371	0.00000307279758299	7.240936×10^{-11}	$3.8078984410 \times 10^{-11}$
10	$9.87645466521414 \times 10^{-7}$	$9.873673916249714 \times 10^{-7}$	0.00000156074174595	4.297712×10^{-11}	$1.674327343 \times 10^{-12}$
11	$1.78543750617433 \times 10^{-7}$	$1.825806691391207 \times 10^{-7}$	0.00000141217846361	2.606582×10^{-12}	$9.103828802 \times 10^{-15}$

5. Conclusions

This paper proposed two iterative methods, namely the ADM and OADM, to solve NODEs of the second and third orders, including the Darcy-Brinkman-Forchheimer moment equation, the Blasius equation, and the Falkner-Skan equation. The study results indicated that the OADM was more accurate and computationally effective compared to the ADM because an optimal control parameter was introduced, which helped to extend the convergence interval and reduce the values of the MER_k. The results revealed that the convergence criterion in the OADM exhibits superior speed and precision relative to the ADM, establishing it as the optimal selection for applications necessitating exact outcomes. The results indicate that when the value of M increases, the accuracy of the solutions improves significantly and the MER_k decreases. Nonetheless, increases in the values of s and F adversely affect accuracy. The findings offer suggestions on adjusting these physical parameters to improve the accuracy of outcomes in the Darcy-Brinkman-Forchheimer moment equation. Besides, the proposed methods have been used to solve various nonlinear equations, including the Blasius equation and the Falkner-Skan equation. Comparison with other numerical methods has proved the superiority of the OADM in accuracy and convergence rate. Comparisons with the BrOM, BOM, and LOM methods indicated that the OADM is remarkably good in all suggested applications. The key contribution of this study is the enhancement of the OADM through the addition of an optimal control parameter. The chosen physical applications are presented to reveal that the optimal method attains fast convergence with a minimal number of iterations, rendering it a useful tool for solving nonlinear systems in practical problems. Future research may include comparison with other numerical and analytical methods in evaluating the optimal method, thereby expanding its application in the boundary value problem and problems related to infinite intervals.

Acknowledgment

The authors express profound gratitude to the reviewers for their insightful remarks and recommendations that enhanced the research.

Conflict of Interest

The authors declare that they have no conflicts of interest.

Funding

There is no financial assistance allocated for the publication preparation.

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