Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2026; 39(2)

Semi-Coessential and Semi-Coclosed Submodules

Raniah Firas Bader[1*], and Nuhad Salim Al-Mothafar[2]
[1,2]Department of Mathematics, Collage of Science, University of Baghdad, Baghdad, Iraq.
*Corresponding Author

Received: 18/August/2025.
Accepted: 7/December/2025.
Published: 20/April/2026
doi.org/10.30526/39.2.4284

Abstract
In this paper, we study and introduce two new concepts in module theory: semi-coessential submodules and semi-coclosed submodules**.** These notions are generalizations of the classical concepts of coessential and coclosed submodules and aim to provide a broader framework for analyzing module structures. We investigate the main properties of these submodules and examine their relationships with coessential and coclosed submodules. Several characteristics associated with these ideas are explored, including conditions under which a submodule can be considered semi-coessential or semi-coclosed. The definitions are formulated in terms of factor modules and the semi-small condition, allowing a systematic approach to understanding their behavior within an R-module. Various propositions and illustrative examples are provided to demonstrate how these new submodules retain essential features of their classical counterparts while offering increased flexibility for structural analysis. The results highlight that semi-coessential submodules generalize the notion of coessential submodules by relaxing certain constraints, whereas semi-coclosed submodules extend coclosed submodules by providing a more flexible framework for identifying maximal submodules with no proper semi-coessential submodules. Overall, these concepts enrich the study of module theory by offering new tools for examining the internal structure of modules and their submodules, paving the way for further research and potential applications in algebraic structures and related areas. Let U and D be submodules of an R-module F such that $U \leq D \leq F$, then U is called semi-coessential submodule of D in F, if $\frac{D}{U} \ll_{semi} \frac{F}{U}$. Moreover, U is said to be semi-coclosed of F, if $\frac{D}{U} \ll_{semi} \frac{F}{U}$ implies that $U = D$.
Keywords: coessential submodules, coclosed submodules, semi-coessential submodules, semi-coclosed submodules.

1. Introduction

In this article, every module is unitary left R-module and every rings have identity. H is a proper submodule of F an R-module is said to be small ($H \ll F$) if, $H + U \neq F,$ for each proper submodule U of F[1-6]. Intuitively, a small submodule is too "weak" or "insufficient" to generate the whole module when combined with any other proper submodule. This property makes small submodules very restrictive and therefore useful in characterizing other types of submodules. A proper submodules U of an R-module F is said to be semi-small ($U \ll_{semi} F$), if $U + H \neq F,$ for any primary submodule H of F[7,8]. Clearly, every small submodule is automatically semi-small, but the converse does not generally hold, which highlights the broader nature of the semi-small condition. When a proper submodule B of F is seen as primary if $rb \in B$, $r \in R$, and $b \in F$ implies that, for any positive number n, either $b \in B$ or $r^n \in [B:F]$, where$[B:F] = \{r \in R: rF \subseteq B\}$[9-15]. This condition generalizes the well-known concept of primary ideals to the more general framework of modules, making it a powerful tool in structural analysis. A submodule D is called a coessential submodule of N in R-module F and denoted by $(D \leq_{ce} N)$, if the quotient $\frac{N}{D}$ is small submodule of $\frac{F}{D}$ [16]. And a submodule D is called a coclosed submodule in R-module F and denoted by $(D \leq_{cc} F)$, if D has no proper coessential submodule in F[17,18]. It is evident from these definitions that small submodules form the basis for studying semi-small submodules of an R-module F. while coessential submodules motivate the study of coclosed ones[19-22]. Several authors have investigated these notions as natural generalizations, providing useful tools to analyze module structures. The concepts of semi-coessential and semi-coclosed submodules, introduced in this work, extend these ideas further, allowing more flexibility and wider applicability in module theory and its algebraic structures.

2. Materials and Methods

2.1. Semi-Coessential Submodules

In this section, we suggest and define a new type of submodules, called semi-coessential submodules, and discuss some of their most important features.

2.1.1. Definition

Let F be an R-module and $E, D \leq F$, such that $E \leq D \leq F$. A submodule E is referred as to be a semi-coessential (s-coessential) submodule of D in F and denoted by $(E \leq_{s.ce} D)$, if $\frac{D}{E} \ll_{semi} \frac{F}{E}$.

2.1.2.Remarks and Examples

- Each coessential submodule is also an s-coessential submodule, since each small submodule is semi-small in general, the convers is not true.
- In Z_6 as a Z-module; a submodule $(\bar{0})$ is not s-coessential of $2Z_6$, since $2Z_6 \simeq \frac{2Z_6}{(\bar{0})}$ is not semi-small of $\frac{Z_6}{(\bar{0})} \simeq Z_6$.
- In Z_4 as a Z-module; $(\bar{0})$ is s-coessential of $2Z_4$, since $\frac{2Z_4}{(\bar{0})} \simeq \{\bar{0}, \bar{2}\} \ll_{semi} \frac{Z_4}{(\bar{0})} \simeq Z_4$ [7].
- Consider Z as a Z-module; $4Z$ is s-coessential of $2Z$, since $\frac{2Z}{4Z} \simeq \{\bar{0}, \bar{2}\} \ll_{semi} \frac{Z}{4Z} \simeq Z_4$ [7].
- Consider Z_8 as a Z-module; a submodule $4Z_8 = \{\bar{0}, \bar{4}\}$ is s-coessential of $2Z_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$, since $\frac{2Z_8}{4Z_8} \simeq \{\bar{0}, \bar{2}\} \ll_{semi} \frac{Z_8}{4Z_8} \simeq Z_4$ by[7].

2.1.3. Proposition

Let F be an R-module, and that E is a semi-small submodule of F, if and only if $(0) \leq_{s.ce} E$ in F.

Proof: Assume E is semi-small of F, since $(0) \leq E \leq F$, therefore by[7] , we result in $\frac{E}{(0)} \ll_{semi} \frac{F}{(0)}$. Hence $(0) \leq_{s.ce} E$ in F. Conversely, let $(0) \leq_{s.ce} E$ in F and $F = E + U$, where U is a primary submodule of F, so $\frac{F}{(0)} = \frac{E+U}{(0)}$, hence $\frac{F}{(0)} = \frac{E}{(0)} + \frac{U}{(0)}$. Since $(0) \leq_{s.ce} E$ in F, then $\frac{E}{(0)} \ll_{semi} \frac{F}{(0)}$. This is a contradiction, since $\frac{U}{(0)}$ is primary submodule of $\frac{F}{(0)}$ and $\frac{E}{(0)} \ll_{semi} \frac{F}{(0)}$. Thus $E + U \neq F$, and hence, $E \ll_{semi} F$.

2.1.4. Proposition

Let E, N, U be submodules of F an R-module with $E \leq N \leq U \leq F$. Then $N \leq_{s.ce} U$ in F, if and only if $\frac{N}{E} \leq_{s.ce} \frac{U}{E}$ in $\frac{F}{E}$.

Proof: Assume $N \leq_{s.ce} U$ in F, thus $\frac{U}{N} \ll_{semi} \frac{F}{N}$. By using the third isomorphism theorem[1], we get $\frac{U}{N} \simeq \frac{\frac{U}{E}}{\frac{N}{E}}$, $\frac{F}{N} \simeq \frac{\frac{F}{E}}{\frac{N}{E}}$, thus $\frac{\frac{U}{E}}{\frac{N}{E}} \ll_{semi} \frac{\frac{F}{E}}{\frac{N}{E}}$ and hence $\frac{N}{E} \leq_{s.ce} \frac{U}{E}$ in $\frac{F}{E}$. Conversely, let $\frac{N}{E} \leq_{s.ce} \frac{U}{E}$ in

$\frac{F}{E}$, therefore $\frac{\frac{U}{E}}{\frac{N}{E}} \ll_{semi} \frac{\frac{F}{E}}{\frac{N}{E}}$ and by using the third isomorphism theorem[1], we get $\frac{U}{N} \simeq \frac{\frac{U}{E}}{\frac{N}{E}} \ll_{semi} \frac{F}{N} \simeq \frac{\frac{F}{E}}{\frac{N}{E}}$ and hence, $\frac{U}{N} \ll_{semi} \frac{F}{N}$. Therefore, $N \leq_{s.ce} U$.

2.1.5. Proposition

Suppose that F is an R-module. Then $D \leq_{s.ce} H$ where $D \leq H \leq A$ and D, H, A are submodules of F. If $D \leq_{s.ce} A$ in F.

Proof: Suppose $\frac{F}{D} = \frac{H}{D} + \frac{K}{D}$, where $\frac{K}{D}$ is a primary submodule of $\frac{F}{D}$. Then $F = H + K$, thus $F = A + K$, since $H \leq A$ so $\frac{F}{D} = \frac{A}{D} + \frac{K}{D}$ by assumption $D \leq_{s.ce} A$ then $\frac{A}{D} \ll_{semi} \frac{F}{D}$ so $\frac{K}{D} = \frac{F}{D}$ is contradiction, therefore, $\frac{H}{D} \ll_{semi} \frac{F}{D}$ and hence, $D \leq_{s.ce} H$.

2.1.6. Lemma

Let U_1 and U_2 be an R-modules with $\theta: U_1 \longrightarrow U_2$ is an R-epimorphism. If L is a semi-small submodule of U_1, such that $\ker \theta \subseteq L$, then $\theta(L)$ is a semi-small submodule of U_2.[7]

2.1.7. Proposition

Let F_1 and F_2 be R-modules with $g: F_1 \longrightarrow F_2$ be an R-epimorphism. If $D \leq_{s.ce} K$ in F_1, then $g(D) \leq_{s.ce} g(K)$ in F_2.

Proof: Define $\alpha: \frac{F_1}{D} \longrightarrow \frac{F_2}{g(D)}$ by $\alpha(x + D) = g(x) + g(D)$, $\forall\, x \in F_1$ it is clear that α is an epimorphism. Since $D \leq_{s.ce} K$ in F_1, thus $\frac{K}{D} \ll_{semi} \frac{F_1}{D}$, by lemma 2.1.6 $\alpha\left(\frac{K}{D}\right) = \frac{g(K)}{g(D)} \ll_{semi} \frac{F_2}{g(D)}$. Hence $g(D) \leq_{s.ce} g(K)$ in F_2.

2.1.8. Lemma

Let U_1 and U_2 be an R-modules with $\theta: U_1 \longrightarrow U_2$ is an R-homomorphism. If L is a primary submodule of U_2, such that $A \subseteq Im\, \theta$, then $\theta^{-1}(L)$ is a primary submodule of U_1.[7]

2.1.9. Proposition

Let F be an R-module and $K \leq H \leq F$. If $F = H + C$, then $F = K + C$, for every submodule C of F. Hence $K \leq_{s.ce} H$ in F.

Proof: Let $\frac{F}{K} = \frac{H}{K} + \frac{L}{K}$, and $\frac{L}{K}$ is a primary submodule in $\frac{F}{K}$, hence $F = H + L$ and by lemma 2.1.8 L is a primary submodule of F, by assumption $F = K + L$ since $K \leq L$, so $F = L$, then $\frac{F}{K} = \frac{L}{K}$ and this is a contradiction, therefore $\frac{H}{K} \ll_{semi} \frac{F}{K}$ and $K \leq_{s.ce} H$ in F.

2.1.10. Remark

The convers is not true of proposition 2.1.9 in general. For example, in Z_{12} as a Z-module, $6Z_{12}$ is s-coessential of $3Z_{12}$ and $2Z_{12} + 3Z_{12} = Z_{12}$ but $6Z_{12} + 2Z_{12} = 2Z_{12} \neq Z_{12}$. To show $\frac{3Z_{12}}{6Z_{12}} \ll_{semi} \frac{Z_{12}}{6Z_{12}}$. The submodule $(\bar{2}), (\bar{3})$ and $(\bar{4})$ are the only primary submodules of Z_{12}, since $\frac{3Z_{12}}{6Z_{12}} \simeq 6Z_{12} \ll_{semi} 2Z_{12} \simeq \frac{Z_{12}}{2Z_{12}}$ so $6Z_{12} \ll_{semi} 2Z_{12}$, hence $6Z_{12}$ is s-coessential of $3Z_{12}$ in Z_{12}.

2.1.11. Proposition

Let F be an R-module, such that $D \leq H \leq F$. If $H = D + W$, and $W \ll_{semi} F$, then $D \leq_{s.ce} H$ in F.

Proof: Let $\frac{C}{D}$ be a primary submodule of $\frac{F}{D}$. We have to show $\frac{H}{D} \ll_{semi} \frac{F}{D}$ and let $\frac{F}{D} = \frac{H}{D} + \frac{C}{D}$, hence $F = H + C$, since $H = D + W$ so $F = (D + W) + C$, implies $F = W + C$, since $W \ll_{semi} F$ and C is a primary submodule of F, so $C = F$ by lemma 2.1.9. Then $\frac{C}{D} = \frac{F}{D}$ and this is a contradiction. Hence $\frac{H}{D} \ll_{semi} \frac{F}{D}$ and $D \leq_{s.ce} H$.

2.1.12. Corollary

Let F be an R-module and suppose that $D \leq H \leq F$. If $F = D + C$ and $C \cap H \ll_{semi} F$, then $D \leq_{s.ce} H$ in F.

Proof: The submodule $H = H \cap F = H \cap (D + C)$ by a module law, we get $H = D + (C \cap H)$, since $C \cap H \ll_{semi} F$, therefore by proposition 2.1.11 $D \leq_{s.ce} H$ in F.

2.2. Semi-Coclosed Submodules

Recall that a submodule K of F is said to be a coclosed submodule (denoted by $K \ll_{cc} F$), if $\frac{K}{H} \ll \frac{F}{H}$ implies that $K = H$.[20]

2.2.1. Definition

An R-module F, a submodule U of F is said to be a semi-coclosed submodule of F (s-coclosed) and denoted by ($U \leq_{s.cc} F$), if $H \leq_{s.ce} U$ for any submodule H of F (i.e. if $\frac{U}{H} \ll_{semi} \frac{F}{H}$) implies that $U = H$.

2.2.2. Remarks and Examples

- Each small submodule is semi-small[7] so every semi-coclosed submodule is coclosed.

Proof: Let D be a s-coclosed submodule of F and $K \leq D$ such that $\frac{D}{K} \ll \frac{F}{K}$, then $\frac{D}{K} \ll_{semi} \frac{F}{K}$ [7]. Since D is a s-coclosed, hence $K = D$. Therefore, D a coclosed submodule of F.

- The convers of (1) in general is not true. For example; Q as Z-module semi-coclosed because each proper submodule of Q is semi-small and it has proper submodules which is not small so that is not coclosed.
- Consider Z_8 as Z-module; a submodule$2Z_8$ is not s-coclosed, because $4Z_8 \leq_{s.ce} 2Z_8$ by remarks and examples 2.2,4 but $4Z_8 \neq 2Z_8$.
- Consider Z as Z-module; a submodule $2Z$ is not s-coclosed, because $\{\bar{0}, \bar{2}\} \simeq \frac{2Z}{4Z} \ll_{semi} \frac{Z}{4Z} \simeq Z_4$, but $4Z \neq 2Z$.
- Consider Z_{12} as Z-module , $6Z_{12}$ is not s-coclosed in Z_{12} since the only submodule of $6Z_{12}$ is $(\bar{0})$, $\frac{6Z_{12}}{(\bar{0})} \simeq 6Z_{12}$ and $\frac{Z_{12}}{(\bar{0})} \simeq Z_{12}$ since $6Z_{12} \ll_{semi} Z_{12}$ and $6Z_{12} \neq (\bar{0})$.

2.2.3. Proposition

Suppose F is an R-module and K, D, C are submodules with $D \leq C \leq F$, then $C \leq_{s.cc} F$ if and only if , $\frac{C}{D} \leq_{s.cc} \frac{F}{D}$.

Proof: Assume $\frac{K}{D} \leq \frac{C}{D}$ and $\frac{K}{D} \leq_{s.ce} \frac{C}{D}$ in $\frac{F}{D}$ by proposition 2.1.4, we get $K \leq_{s.ce} C$ in F and since $C \leq_{s.cc} F$ then $K = H$ and hence $\frac{K}{D} = \frac{H}{D}$.

Conversely; Suppose $K \leq_{s.ce} C$ in F by proposition 2.1.4, we get $\frac{K}{D} \leq_{s.ce} \frac{C}{D}$ in $\frac{F}{D}$ and since $\frac{C}{D} \leq_{s.cc} \frac{F}{D}$ then $\frac{K}{D} = \frac{C}{D}$ and hence $K = C$.

2.2.4. Proposition

Suppose F is an R-module, a submodule K is a non-zero of F, if $K \leq_{s.cc} F$ then K is not semi-small in F.

Proof: Suppose $K \ll_{semi} F$ and $K \leq_{s.cc} F$. Since $(0) \leq K$ and $K \simeq \frac{K}{(0)} \ll_{semi} \frac{F}{(0)} \simeq F$. Then $K = (0)$ which is a contradiction. Therefore, K is not semi-small in .

2.2.5. Proposition

Suppose F is an R-module and $K_1 \leq K_2 \leq F$ and if $K_1 \leq_{s.cc} F$, then $K_1 \leq_{s.cc} K_2$.

Proof: Assume $D \leq K_1$ such that $\frac{K_1}{D} \ll_{semi} \frac{K_2}{D} \leq \frac{F}{D}$, so by[7] $\frac{K_1}{D} \ll_{semi} \frac{F}{D}$. Since $K_1 \leq_{s.cc} F$ implies that $D = K_1$, therefore $K_1 \leq_{s.cc} K_2$.

Recall that a module F of an R-module, if each proper submodule is semi-small in F then F is called s-hollow module[23].

2.2.6. Proposition

For each non zero s-coclosed submodule of an s-hollow module is s-hollow.

Proof: let F be a module is an s-hollow and H is s-coclosed in F and let B be a proper submodule of H and C be a primary submodule of H such that $H = B + C$. Since F is s-hollow[23]

. $\frac{F}{C}$ is s-hollow. If $\frac{H}{C}$ is a proper submodule of $\frac{F}{C}$ so $\frac{H}{C} \ll_{semi} \frac{F}{C}$, but H is s-coclosed, then $H = C$ this is a contradiction so $B \ll_{semi} H$. Thus H is s-hollow.
Remember that a module F of an R-module, if each proper submodule of F is prime then F is said to be a fully prime module [24]. And if each proper submodule of F is primary then F is said to be a fully primary module[25].

3. Examples and Propositions

- Every fully prime module is fully primary module; in general, is not true the convers. For example; in Z_8 as Z-module is fully primary module but isn't fully prime module.
- Consider Z_{12} as Z-module is not fully primary, because $6Z_{12}$ is not primary submodule of Z_{12}.
- In Z_6 as Z-module is fully primary, because $2Z_6$ and $3Z_6$ are primary submodules of Z_6.

3.1. Proposition

Let F be an R-module is a fully primary module and $K_1 \leq K_2 \leq F$. If $\frac{K_2}{K_1} \leq_{s.cc} \frac{F}{K_1}$ and $K_1 \ll_{semi} K_2$, then $K_2 \leq_{s.cc} F$.

Proof: Suppose that $D \leq K_2$ such that $\frac{K_2}{D} \ll_{semi} \frac{F}{D}$, and let $\pi: F \longrightarrow \frac{F}{K_1}$ is the natural epimorphism, since $D \leq_{s.ce} K_2$, then by proposition 2.1.7 $\frac{D+K_1}{K_1} \leq_{s.ce} \frac{K_2}{K_1}$ implies that $\frac{\frac{K_2}{K_1}}{\frac{D+K_1}{K_1}} \ll_{semi} \frac{\frac{F}{K_1}}{\frac{D+K_1}{K_1}}$. But $\frac{K_2}{K_1} \leq_{s.cc} \frac{F}{K_1}$, so $\frac{D+K_1}{K_1} = \frac{K_2}{K_1}$ and hence $D + K_1 = K_2$. Since F is a fully primary module, D is primary submodule of F and $D \leq K_2$ and K_2 is primary in F then D is primary submodule of K_2, but K_1 is semi-small of K_2, thus $K_2 = D$. Therefore $K_2 \leq_{s.cc} F$.

3.2. Proposition

Let $F = F_1 \oplus F_2$ be an R-module. If $K \ll_{s.cc} F_1$, then $K \ll_{s.cc} F$.

Proof: Let $H \leq K$ such that $\frac{K}{H} \ll_{semi} \frac{F}{H} = \frac{F_1 \oplus F_2}{H}$, then $\frac{K}{H} \ll_{semi} \frac{F_1}{H} \oplus \frac{F_2 \oplus H}{H}$ by[7] $\frac{K}{H} \ll_{semi} \frac{F_1}{H}$. Since $K \ll_{s.cc} F_1$, then $H = K$ and $K \ll_{s.cc} F$.

4. Results and Discussion

Generalizing classical submodule concepts allows broader exploration of module properties, and semi-properties maintain essential structural elements while providing more flexibility, and the results offer a foundation for studying weaker module structures and broader applications in algebra. Examples showed how semi-properties can be distinguished from their classical counterparts. Future research can build upon these concepts to explore deeper relationships in module theory.

5. Conclusion

The concept of semi-coessential submodules was introduced as a generalization of coessential submodules, and the concept of semi-coclosed submodules was introduced as a generalization of coclosed submodules, and various propositions and examples were used to explore the properties and relationships of these new submodules. It was demonstrated that semi-coessential and semi-coclosed submodules retain some core traits while offering broader applicability, and findings provide a foundation for future research in module theory and algebraic structures.

Acknowledgment

The quality and clarity of the work were greatly enhanced by the incisive suggestions and constructive criticism provided by the anonymous referees, for which we are truly grateful.

Conflict of Interest

The authors claim that there are no conflicts of interest.

Funding

The authors confirm that the research was carried out independently and did not benefit from any financial support.

References

1. Nebiyev C, Ökten HH. Some properties of r-small submodules. Erzincan Univ J Sci Technol. 2022;15(3):996-1001. https://doi.org/10.18185/erzifbed.876557
2. Chaturvedı AK, Kumar N. On modules with chain condition on non-small submodules. Int Electron J Algebra. 2023;33(33):109-124. https://doi.org/10.24330/ieja.1195509
3. Neama SM, Alwan AH. A new approach to multiplication modules via e-small submodules. J Math Probl Equ Stat. 2025; 6(2): 524-528. https:/doi.org/10.22271/math.2025.v6.i2c.253
4. Ozdemir HB, Nebiyev C. Strongly ⨁-g-radical supplemented modules. Miskolc Math Notes. 2023;24(2):981-987. https://doi.org/10.18514/MMN.2023.4029
5. Hanoon WH, Abboodi AA. Large-maximal small submodules. J Interdiscip Math. 2023;26(6):1043-1052. https://doi.org/10.47974/JIM-1604
6. Mohammed OB, Ghawi TY. About e-gH modules. J Al-Qadisiyah Comput Sci Math. 2023;15(1):121-126. https://doi.org/10.29304/jqcm.2023.15.1.1174
7. Mijbass AS, Abdullah NK. Semi-small submodules. Tikrit J Pure Sci. 2011;16(1):104-107.
8. Diallo AD, Diop PC, Barry M. On s-quasi-dedekind modules. J Math Res. 2017;9(5):97-107. https://doi.org/10.5539/jmr.v9n5p97
9. Celikel EY, Khashan HA. On weakly S-primary submodules. Filomat. 2023;37(8):2503-2516. https://doi.org/10.2298/FIL2308503Y
10. Hamoda M. On n-absorbing primary submodules. Novi Sad J Math. 2022;52(1):143-154. https://doi.org/10.30755/NSJOM.11073
11. Celikel EY. 1-absorbing primary submodules. Ann Univ Ovidius Constanta. 2021;29(3):285-296. https://doi.org/10.2478/auom-2021-0045
12. Naderi MH, Jahani-Nezhad R. On primary and classical primary submodules. Southeast Asian Bulletin of Mathematics. 2018;42(3): 401-410.
13. Jafari A, Baziar M, Safaeeyan S. A generalization of primary ideals and strongly prime submodules. Rev Union Mat Argent. 2021;62(2):423-432. https://doi.org/10.33044/revuma.1783
14. Ersoy BA, Tekir Ü, Kaya E, Bolat M, Koç S. On ϕ-δ-primary Submodules. Iran J Sci Technol Trans A Sci. 2022;46(2):421-427. https://doi.org/10.1007/s40995-022-01271-z
15. Khalf MF, Radhi OA. Weakly semi-primary submodules. Int J Nonlinear Anal Appl. 2022;13(2):185-190.
16. Shahoodh MK, Ali OK. T-Closed-Coessential and T-Closed-Coclosed Submodules. Int J Anal Appl. 2025; 28(23):299. https://doi.org/10.28924/2291-8639-23-2025-299
17. Büyükaşık E, Demir Ö, Diril M. On simple-direct modules. Commun Algebra. 2021;49(2):864-876. https://doi.org/10.1080/00927872.2020.1821207
18. Kamil EM, Al-Aeashi S. γ-coclosed dual rickart modules. AIP Conf Proc. 2023;2591(1):050025. https://doi.org/10.1063/5.0120431
19. Talebi Y, Amoozegar T. Strongly FI-lifting modules. Int Electron J Algebra. 2008;3(3):75-82.
20. Wisbauer R. Foundations of module and ring theory. London: Routledge; 2018.
21. Tütüncü DK, Toksoy SE. Absolute co-supplement and absolute co-coclosed modules. Hacet J Math Stat. 2013;42(1):67-79.
22. Talebi YA, Talaee BE. On δ-coclosed submodules. Far East J Math Sci. 2009;35(1):19-31.
23. Al Hakeem MB, Al-Mothafar NS, Haghjooyan M. Semi-small compressible modules and semi-small retractable modules. Ibn Al-Haitham J Pure Appl Sci. 2023;36(4):407-413. https://doi.org/10.30526/36.4.3156
24. Al-Silaykhee AM, Al-Aeashi SN. Primely compressible module relative to a submodule. J Phys Conf Ser. 2021;1963(1):012133. https://doi.org/10.1088/1742-6596/1963/1/012133
25. Nikseresht A, Sharif H. Fully primary modules and some variations. J Algebra Relat Top. 2013;1(1):1-7.