



New Weibull X-Lindley distribution: Structure and Properties

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Abstract

This paper aims to introduce a novel univariate probability distribution called the Weibull X-Lindley distribution (W-XLD), designed to be more flexible in modeling data with different patterns. The core idea behind this distribution arose to address some of the limitations of traditional distributions by introducing a form factor to increase flexibility to the proposed distributions. The technique used to construct W-XLD relies on multiplying the survival functions of the X-Lindley and Weibull distributions to find the W-XLD survival function. Based on this, the statistical functions of W-XLD were defined, including the cumulative distribution function (cdf), the probability density function (pdf), and the risk function. The shapes of the distribution functions for various conditional data and parameter values were discussed. The distribution parameters are two: the shape parameter (β) and the scale parameter (θ). The properties of the density and risk functions at the boundaries of the random variable domain were examined, based on finding their boundary values. Furthermore, the basic and statistical characteristics were extracted, analyzed, given, and studied, including pattern, moments, kurtosis, skewness, variance, generated moments, quantiles, mean time to failure, and segmentation.

Keywords: Weibull X-Lindley, X-Lindley, Characteristic Function, Mean time to failure, Quantile function.

1.Introduction

Statistics is that has seen widespread development across various fields, including Medicine, Engineering, and Economics. Therefore, statisticians and scientists have resorted to creating new distributions to provide flexibility when dealing with complex distributions. Furthermore, problems encountered in some complex distributions will be addressed, and the method of multiplying survival functions is one of several ways to construct a new distribution, it's credited with pioneering such constructions¹. A researcher also introduced a combination, which is a mixture of exponential and Lindley distributions. The X-Lindley distribution (XLD), a one-parameter distribution, was created by combining the exponential and Lindley distribution^{2,3}, following are (c.d.f) and the (p.d.f.) of (XLD):

$$F_{XL}(y) = 1 - \frac{1}{2}(2 + \theta y)e^{-\theta y}, y > 0, \theta > 0 \quad (1)$$

$$f_{XL}(y) = \frac{\theta}{2}(1 + \theta y)e^{-\theta y} \quad (2)$$

And survival (reliability) function defined as:

$$S_{XL}(y) = \frac{1}{2}(2 + \theta y)e^{-\theta y} \quad (3)$$

Another important distribution is Weibull distribution, a one-parameter distribution with broad applicability. This distribution is significant due to its broad applicability. The (c.d.f.), (p.d.f.) and the survival function are:

$$F_W(z) = 1 - e^{-z^\beta}, z > 0, \beta > 0 \quad (4)$$

$$f_W(z) = \beta z^{\beta-1} e^{-z^\beta} \quad (5)$$

$$S_W(z) = e^{-z^\beta} \quad (6)$$

Also we presented our new model by multiplying the survival functions of the X-Lindley distribution with the Weibull distribution's survival function, giving rise to a novel distribution known as the Weibull X-Lindley (W-XLD) distribution. Furthermore, researchers have developed distributions such as the Lindley-Weibull distribution^{4,5}. The Weibull distribution was the first to use survival function multiplication, as seen in the Weibull-exponential distribution⁶, the Weibull-Pareto distribution⁷, the Beta-Weibull distribution⁸, the Topp-Leone Weibull-Lomax distribution⁹, and the Weibull-Rayleigh distribution¹⁰. Others have introduced new distributions such as^{11,12}. Still others have used different methods such as weighted addition and structure to create new distributions^{4,5,13,14}, the modified Rayleigh distribution¹⁵, the weighted exponential distribution¹⁶, the Shanker Weibull distribution¹⁷, the variance-gamma distribution¹⁸. Other models are generalizations of families that were submitted to generate distributions¹⁹⁻²⁵. In this study, we introduce a novel statistical distribution, termed the Weibull X-Lindley distribution. Subsequently, we will detail the derivation and analysis of all its distributional characteristics, and we will demonstrate the graphing of distribution functions, and we will discuss this in detail in this paper.

2. Materials and Methods

2.1. Structure of New Weibull X-Lindley distribution:

Let y and z be random variables and $x = \min\{y, z\}$

$$S(x) = P_r(X > x) = P_r(\min\{y, z\} > x),$$

From **Equations 3 and 6**, and since Z and Y are independent random variables then:

$$\begin{aligned} S_{(WXL)}(x) &= P_r(Z > x) \cdot P_r(Y > x) = (1 - P_r(Z \leq x)) \cdot (1 - P_r(Y \leq x)) \\ &= (1 - F_W(z)) \cdot (1 - F_{XL}(y)) = S_W(z) \cdot S_{XL}(y) \end{aligned}$$

$$S_{WXL}(x) = \frac{1}{2} (2 + \vartheta x) e^{-(\vartheta x + x^\beta)} \quad (7)$$

For $x > 0$, and $\vartheta, \beta > 0$ are the scale and shape parameters respectively. The (c.d.f) of the W-XLD is:

$$F_{WXL}(x) = 1 - \frac{1}{2} (2 + \vartheta x) e^{-(\vartheta x + x^\beta)}, x > 0, \beta, \vartheta > 0 \quad (8)$$

Therefore, the pdf of the new (WXL) is:

$$f_{WXL}(x) = \frac{1}{2} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta \beta x^\beta) e^{-(\vartheta x + x^\beta)} \quad (9)$$

Equation 9 is represented a p.d.f of W-XLD, since, $\beta > 0, \vartheta > 0$, and $x > 0$, which implies that $f(x) > 0$.

Now, to prove that $\int_0^\infty f(x) dx = 1$,

$$\begin{aligned} \int_0^\infty f(x) dx &= \int_0^\infty \frac{1}{2} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta \beta x^\beta) e^{-(\vartheta x + x^\beta)} dx \\ \int_0^\infty f(x) dx &= \lim_{y \rightarrow \infty} \int_0^y \frac{1}{2} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta \beta x^\beta) e^{-(\vartheta x + x^\beta)} dx \\ &= \lim_{y \rightarrow \infty} \left(1 - \frac{1}{2} (2 + \vartheta y) e^{-(\vartheta y + y^\beta)} \right) \Big|_0^y = 1 - \frac{1}{2} \lim_{y \rightarrow \infty} \left(\frac{(2 + \vartheta y)}{e^{(\vartheta y + y^\beta)}} \right) \end{aligned}$$

By using L'Hopital's - Role for the above limit,

$$\int_0^\infty f(x) dx = 1 - \frac{1}{2} \lim_{y \rightarrow \infty} \left(\frac{\vartheta}{(\vartheta + \beta y^{\beta-1}) e^{(\vartheta y + y^\beta)}} \right) = 1 - \frac{\vartheta}{\infty} = 1$$

As previously mentioned, the hazard function can be expressed in the following way:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\frac{1}{2}(\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta)e^{-(\vartheta x + x^\beta)}}{\frac{1}{2}(2 + \vartheta x)e^{-(\vartheta x + x^\beta)}}$$

$$h(x) = \frac{(\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta)}{(2 + \vartheta x)} \quad (10)$$

2.2 The shapes of the new Weibull X-Lindley distribution:

It is important to understand the shape of the X-WLD and create tables that illustrate how the distribution works at specific values, calculating this mathematically so that we can identify the new distribution at certain points ($x = \infty, x = 0$).

$$\lim_{x \rightarrow \infty} (f(x)) = \frac{1}{2} \lim_{x \rightarrow \infty} \left((\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta) e^{-(\vartheta x + x^\beta)} \right)$$

$$\lim_{x \rightarrow \infty} (f(x)) = \frac{1}{2} \lim_{x \rightarrow \infty} \left(\frac{(\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta)}{e^{(\vartheta x + x^\beta)}} \right)$$

By using L'Hopital's-Rule for the above limit,

$$\lim_{x \rightarrow \infty} (f(x)) = \frac{1}{2} \lim_{x \rightarrow \infty} \left(\frac{\vartheta^2 + 2\beta(\beta - 1)x^{\beta-2} + \vartheta\beta^2 x^{\beta-1}}{(\vartheta + \beta x^{\beta-1})e^{(\vartheta x + x^\beta)}} \right)$$

Note: As we continue to differentiate the variable x , the result of the numerator will be constant, and the limit of the denominator will be equal to ∞ as ($x \rightarrow \infty$) when the limit of the function will be a constant value on x , which is equal to zero.

$$\lim_{x \rightarrow \infty} (f(x)) = 0$$

$$\lim_{x \rightarrow 0} (f(x)) = \frac{1}{2} \lim_{x \rightarrow 0} \left((\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta) e^{-(\vartheta x + x^\beta)} \right)$$

$$\lim_{x \rightarrow 0} (f(x)) = \frac{1}{2} \lim_{x \rightarrow 0} \left(\frac{(\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta)}{e^{(\vartheta x + x^\beta)}} \right) = \vartheta + 2\beta + \vartheta\beta$$

$$\lim_{x \rightarrow \infty} (h(x)) = \lim_{x \rightarrow \infty} \left(\frac{(\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta)}{(2 + \vartheta x)} \right)$$

By using L'Hopital's - Role for the above limit,

$$\lim_{x \rightarrow \infty} (h(x)) = \lim_{x \rightarrow \infty} \left(\frac{\vartheta^2 + 2\beta(\beta - 1)x^{\beta-2} + \vartheta\beta^2 x^{\beta-1}}{\vartheta} \right) = \frac{\infty}{\vartheta} = \infty, \vartheta > 0$$

$$\lim_{x \rightarrow 0} (h(x)) = \lim_{x \rightarrow 0} \left(\frac{(\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta)}{(2 + \vartheta x)} \right) = \frac{\vartheta + 2\beta + \vartheta\beta}{2}$$

Figure 1 shows the behavior of the cumulative distribution function (CDF) for the W-XLD at different values of the parameters θ and β , highlighting the effect of their variation on the probability accumulation rate. **Figure 2** shows the behavior of the probability density function (PDF) for the W-XLD at different values of the parameters θ and β , demonstrating their effect on the curve shape, mode position, and degree of dispersion. **Figure 3** shows the behavior of the survival function for the W-XLD at different values of the parameters θ and β , illustrating their effect on the rate of decrease in the probability of survival as x increases. **Figure 4** shows the behavior of the hazard function of the W-XLD at different values of the two parameters θ and β , showing their effect on the pattern of change of the hazard rate with increasing x . The probability density function (PDF), the cumulative distribution function (CDF), the survival function (SF), and the risk function (HF) were derived. The results showed that the risk function can take multiple forms (increasing, decreasing, single-peak, or pseudo-trough) depending on the parameter values, thus enhancing the model's ability to represent age data, reliability, and nonlinear failure rate behavior.

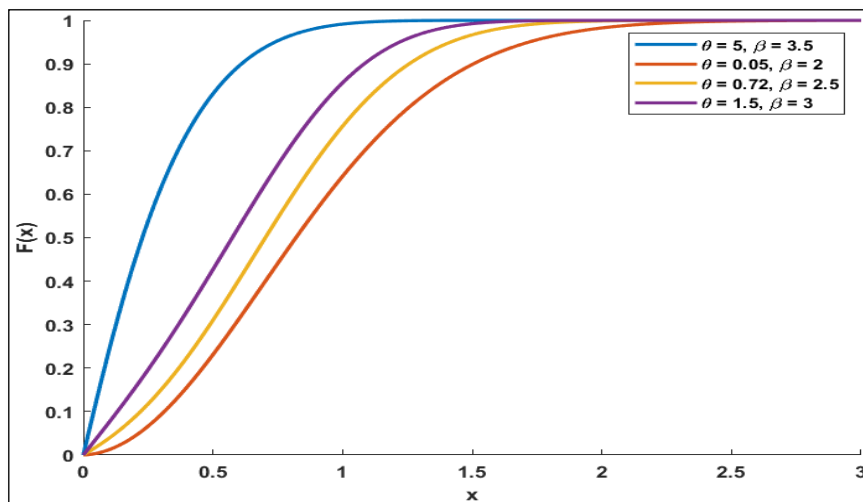


Figure 1. The shape of cdf for new Weibull X-Lindley distribution with different values of ϑ and β .

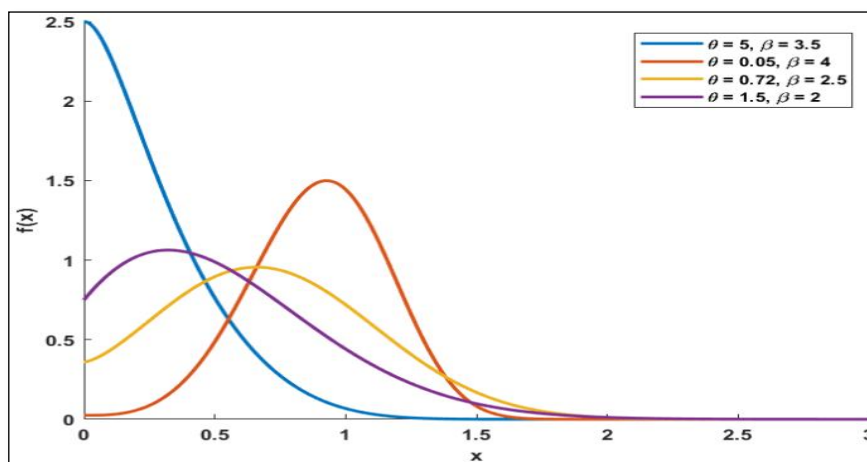


Figure 2. The shape of pdf for new Weibull X-Lindley distribution with different values of ϑ and β .

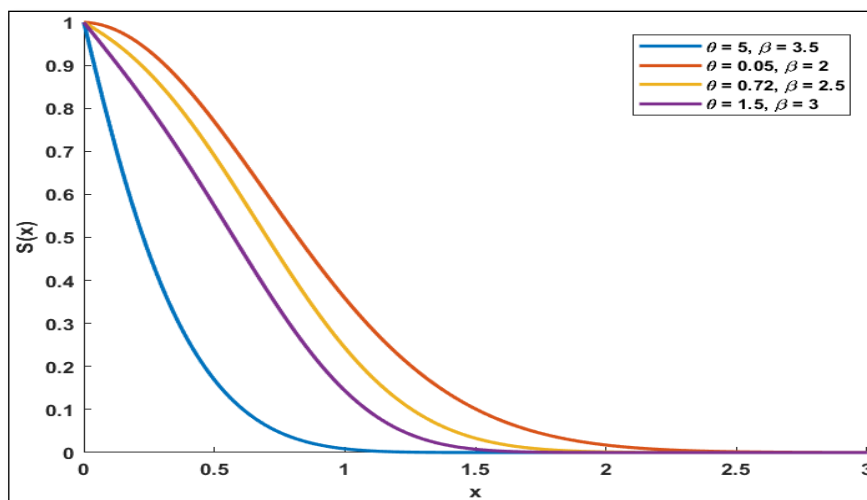


Figure 3. The shape of $S(x)$ for new Weibull X-Lindley distribution with different values of ϑ and β

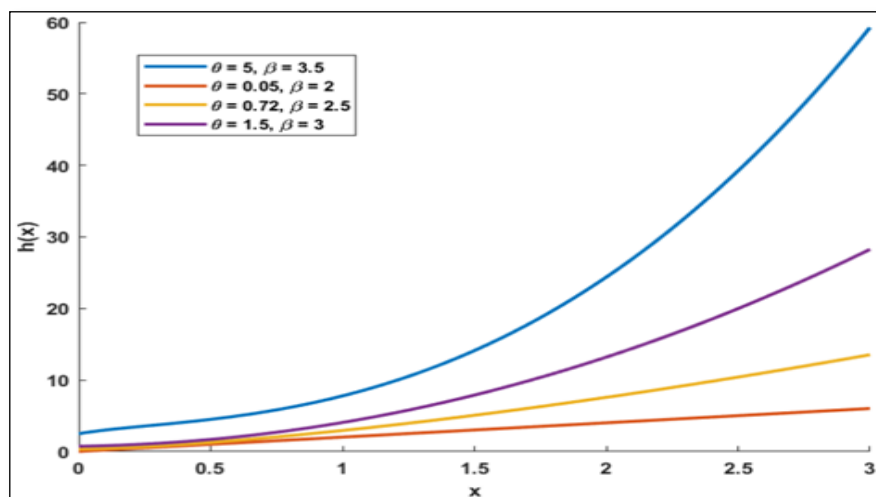


Figure 4. The shape of $h(x)$ for new Weibull X-Lindley distribution with different values of ϑ and β .

3. Mathematical and statistical properties of X-WLD:

3.1 The Mode

The mode is identified as the value that appears most often, and the mode X-WLD is derived through the following process:

$$f_{WXL}(x) = \frac{1}{2} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta) e^{-(\vartheta x + x^\beta)}$$

$$\frac{\partial f(x)}{\partial x} = \frac{1}{2} \left((\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta) (-(\vartheta + \beta x^{\beta-1}) e^{-(\vartheta x + x^\beta)} + e^{-(\vartheta x + x^\beta)} (\vartheta^2 + 2\beta(\beta-1)x^{\beta-2} + \vartheta\beta^2 x^{\beta-1})) \right)$$

$$\frac{\partial f(x)}{\partial x} = 0$$

$$\frac{1}{2} \left((\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta) (-(\vartheta + \beta x^{\beta-1}) e^{-(\vartheta x + x^\beta)} + e^{-(\vartheta x + x^\beta)} (\vartheta^2 + 2\beta(\beta-1)x^{\beta-2} + \vartheta\beta^2 x^{\beta-1})) \right) = 0$$

Since that $e^{-(\vartheta x + x^\beta)} \neq 0$ for each selected value of ϑ, β , and x , this leads to the following:

$$\left((\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^\beta) (-(\vartheta + \beta x^{\beta-1}) + (\vartheta^2 + 2\beta(\beta-1)x^{\beta-2} + \vartheta\beta^2 x^{\beta-1})) \right) = 0$$

The above equation is a nonlinear equation, and finding solutions for it (x) is done using numerical methods for solving nonlinear equations. To understand the nature of the solutions to find the materials, we choose default values for the parameter.

If $\beta = 1$

$$((\vartheta + \vartheta^2 x + 2 + \vartheta x) (\vartheta^2 - 1)) = 0$$

$$\vartheta + \vartheta^2 x + 2 + \vartheta x = 0$$

$$x(\vartheta^2 + \vartheta) = -(\vartheta + 2)$$

$$x = \frac{-(\vartheta + 2)}{(\vartheta^2 + \vartheta)} \quad (11)$$

3.2 Quantile Function

Let X be a random variable following distribution (W-XLD); then the quantity function is defined as

$$x = Q(u) = F^{-1}(u)$$

$$\text{let } u = 1 - \frac{1}{2} (2 + \vartheta x) e^{-(\vartheta x + x^\beta)}, u \in (0,1)$$

$$(2 + \vartheta x) e^{-(\vartheta x + x^\beta)} = 2(1 - u)$$

When $\beta = 1$, and let $y = 2 + \vartheta x$, then

$$ye^{\frac{-(\vartheta+1)(y-2)}{\vartheta}} = 2(1-u)$$

$$ye^{\frac{-(\vartheta+1)y}{\vartheta}} = 2(1-u)e^{\frac{-2(\vartheta+1)}{\vartheta}}$$

Let $z = \frac{-(\vartheta+1)y}{\vartheta}$, multiply both sides by $\left(\frac{-(\vartheta+1)}{\vartheta}\right)$, and apply Lambert function $\mathcal{W}$:

$$ze^z = \frac{-2(\vartheta+1)(1-u)}{\vartheta} e^{\frac{-2(\vartheta+1)}{\vartheta}}$$

$$z = \mathcal{W}\left(\frac{-2(\vartheta+1)(1-u)}{\vartheta} e^{\frac{-2(\vartheta+1)}{\vartheta}}\right)$$

$$x = \frac{-1}{\vartheta+1} \mathcal{W}_{-1}\left(\frac{-2(\vartheta+1)(1-u)}{\vartheta} e^{\frac{-2(\vartheta+1)}{\vartheta}}\right) - \frac{2}{\vartheta} \quad (12)$$

Where $(\mathcal{W}_{-1})$ is (-1) Lambert function with noting that $x > 0$.

A special case of **Equation 12** is represented by the median when $(u=0.5)$.

$$x = \frac{-1}{\vartheta+1} \mathcal{W}_{-1}\left(\frac{-(\vartheta+1)}{\vartheta} e^{\frac{-2(\vartheta+1)}{\vartheta}}\right) - \frac{2}{\vartheta} \quad (13)$$

3.3. Moments about the origin

Moments play a crucial role in the properties of any distribution, as medium, variance, skewness, and kurtosis are determined through the use of moments. As follows:

$$\mu'_r = E(x^r) = \int_0^{\infty} x^r f(x) dx$$

$$\mu'_r = \int_0^{\infty} x^r \frac{1}{2} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^{\beta}) e^{-(\vartheta x + x^{\beta})} dx$$

$$\mu'_r = \frac{1}{2} \int_0^{\infty} (\vartheta x^r + \vartheta^2 x^{r+1} + 2\beta x^{\beta+r-1} + \vartheta\beta x^{\beta+r}) e^{-(\vartheta x + x^{\beta})} dx$$

$$\mu'_r = \frac{1}{2} \left(\vartheta \int_0^{\infty} x^r e^{-(\vartheta x + x^{\beta})} dx + \vartheta^2 \int_0^{\infty} x^{r+1} e^{-(\vartheta x + x^{\beta})} dx + 2\beta \int_0^{\infty} x^{\beta+r-1} e^{-(\vartheta x + x^{\beta})} dx + \vartheta\beta \int_0^{\infty} x^{\beta+r} e^{-(\vartheta x + x^{\beta})} dx \right)$$

$$= \frac{1}{2} \left(\vartheta \int_0^{\infty} x^r e^{-\vartheta x} e^{-x^{\beta}} dx + \vartheta^2 \int_0^{\infty} x^{r+1} e^{-\vartheta x} e^{-x^{\beta}} dx + 2\beta \int_0^{\infty} x^{\beta+r-1} e^{-\vartheta x} e^{-x^{\beta}} dx + \vartheta\beta \int_0^{\infty} x^{\beta+r} e^{-\vartheta x} e^{-x^{\beta}} dx \right)$$

$$e^{-x^{\beta}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^{i\beta}$$

$$\mu'_r = \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\vartheta \int_0^{\infty} x^{r+i\beta} e^{-\vartheta x} dx + \vartheta^2 \int_0^{\infty} x^{i\beta+r+1} e^{-\vartheta x} dx + 2\beta \int_0^{\infty} x^{\beta(i+1)+r-1} e^{-\vartheta x} dx + \vartheta\beta \int_0^{\infty} x^{\beta(i+1)+r} e^{-\vartheta x} dx \right)$$

let $y = \vartheta x \Rightarrow x = \frac{y}{\vartheta}$, and $dx = \frac{dy}{\vartheta}$, if $x = 0 \rightarrow y = 0$ and if $x = \infty \rightarrow y = \infty$

$$\begin{aligned} \mu'_r &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\vartheta \int_0^{\infty} \left(\frac{y}{\vartheta}\right)^{r+i\beta} e^{-y} \frac{dy}{\vartheta} + \vartheta^2 \int_0^{\infty} \left(\frac{y}{\vartheta}\right)^{i\beta+r+1} e^{-y} \frac{dy}{\vartheta} \right. \\ &\quad \left. + 2\beta \int_0^{\infty} \left(\frac{y}{\vartheta}\right)^{\beta(i+1)+r-1} e^{-y} \frac{dy}{\vartheta} + \vartheta\beta \int_0^{\infty} \left(\frac{y}{\vartheta}\right)^{\beta(i+1)+r} e^{-y} \frac{dy}{\vartheta} \right) \\ \mu'_r &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{1}{\vartheta^{r+i\beta}} \int_0^{\infty} y^{(r+i\beta)-1} e^{-y} dy + \frac{1}{\vartheta^{i\beta+r}} \int_0^{\infty} y^{(i\beta+r)-1} e^{-y} dy \right. \\ &\quad \left. + \frac{2\beta}{\vartheta^{\beta(i+1)+r}} \int_0^{\infty} y^{(\beta(i+1)+r)-1} e^{-y} dy + \frac{\beta}{\vartheta^{\beta(i+1)+r}} \int_0^{\infty} y^{(\beta(i+1)+r)-1} e^{-y} dy \right) \\ \mu'_r &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\Gamma_{r+i\beta+1}}{\vartheta^{r+i\beta}} + \frac{\Gamma_{i\beta+r+2}}{\vartheta^{i\beta+r}} + \frac{2\beta\Gamma_{\beta(i+1)+r}}{\vartheta^{\beta(i+1)+r}} + \frac{\beta\Gamma_{\beta(i+1)+r+1}}{\vartheta^{\beta(i+1)+r}} \right) \end{aligned} \quad (14)$$

We will give special cases for **Equation 14**, in addition to the variance, skewness (C.S) and kurtosis(C.K).

$$\mu'_1 = \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\Gamma_{i\beta+2}}{\vartheta^{1+i\beta}} + \frac{\Gamma_{i\beta+3}}{\vartheta^{i\beta+1}} + \frac{2\beta\Gamma_{\beta(i+1)+1}}{\vartheta^{\beta(i+1)+1}} + \frac{\beta\Gamma_{\beta(i+1)+2}}{\vartheta^{\beta(i+1)+1}} \right) \quad (15)$$

$$\mu'_2 = \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\Gamma_{i\beta+3}}{\vartheta^{2+i\beta}} + \frac{\Gamma_{i\beta+4}}{\vartheta^{i\beta+2}} + \frac{2\beta\Gamma_{\beta(i+1)+2}}{\vartheta^{\beta(i+1)+2}} + \frac{\beta\Gamma_{\beta(i+1)+3}}{\vartheta^{\beta(i+1)+2}} \right) \quad (16)$$

$$\mu'_3 = \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\Gamma_{i\beta+4}}{\vartheta^{3+i\beta}} + \frac{\Gamma_{i\beta+5}}{\vartheta^{i\beta+3}} + \frac{2\beta\Gamma_{\beta(i+1)+3}}{\vartheta^{\beta(i+1)+3}} + \frac{\beta\Gamma_{\beta(i+1)+4}}{\vartheta^{\beta(i+1)+3}} \right) \quad (18)$$

$$\mu'_4 = \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\Gamma_{i\beta+5}}{\vartheta^{4+i\beta}} + \frac{\Gamma_{i\beta+6}}{\vartheta^{i\beta+4}} + \frac{2\beta\Gamma_{\beta(i+1)+4}}{\vartheta^{\beta(i+1)+4}} + \frac{\beta\Gamma_{\beta(i+1)+5}}{\vartheta^{\beta(i+1)+4}} \right) \quad (19)$$

$$var(x) = \mu'_2 - (\mu'_1)^2 \quad (20)$$

3.4 Coefficients of Skewness and Kurtosis

The coefficients for skewness (C.S) and kurtosis (C.K) can be calculated using the formulas provided below:

$$SK = \frac{\mu'_3}{\sqrt{(\mu'_2)^3}} \quad (21)$$

$$KU = \frac{\mu'_4}{(\mu'_2)^2} \quad (22)$$

Table. 1 shows that the properties of the W-XLD are clearly affected by variations in its two parameters, ϑ and β . The first four moments reflect the distribution's flexibility in representing different degrees of central tendency and dispersion. The variance values range from low to

moderate, indicating the distribution's ability to model data with relatively limited variability. Furthermore, the $\mathcal{SK}$ values are positive in all cases, indicating that the distribution is right-skewed, a behavior appropriate for age and failure time data. In contrast, the $\mathcal{KU}$ values vary between negative and positive, suggesting that the distribution may be flatter or steeper than a normal distribution depending on the parameter values.

Table 1. 1st - 4th moments, skewness, kurtosis, variance

ϑ	β	μ'_1	μ'_2	μ'_3	μ'_4	var	$\mathcal{SK}$	$\mathcal{KU}$
2.5	1.5	0.3348	0.1787	0.1151	0.0826	0.0666	1.5231	-0.4151
1.25	0.5	0.2254	0.1224	0.0825	0.0618	0.0716	1.9262	1.1226
3.5	0.05	0.1287	0.0660	0.0421	0.0302	0.0495	2.4815	3.9183
0.5	2.5	0.4199	0.2917	0.2211	0.1768	0.1154	1.4030	-0.9229
0.25	0.75	0.2331	0.1341	0.0929	0.0707	0.0797	1.8919	0.9339
0.01	1.25	0.3018	0.1913	0.1382	0.1076	0.1002	1.6522	-0.0593

3.5 Characteristic Function

The characteristic function in statistics is the Fourier transform, and it can be used to find the moments of a random variable, as follows:

$$\phi_x(it) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x) dx = \int_0^{\infty} e^{itx} \frac{1}{2} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^{\beta}) e^{-(\vartheta x + x^{\beta})} dx$$

$$\phi_x(it) = \frac{1}{2} \int_0^{\infty} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta\beta x^{\beta}) e^{-(\vartheta - it)x + x^{\beta}} dx$$

$$e^{-x^{\beta}} = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} x^{\beta v}$$

$$\phi_x(it) = \frac{1}{2} \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} \left(\vartheta \int_0^{\infty} x^{\beta v} e^{-(\vartheta - it)x} dx + \vartheta^2 \int_0^{\infty} x^{\beta v + 1} e^{-(\vartheta - it)x} dx \right. \\ \left. + 2\beta \int_0^{\infty} x^{\beta + \beta v - 1} e^{-(\vartheta - it)x} dx + \vartheta\beta \int_0^{\infty} x^{\beta + \beta v} e^{-(\vartheta - it)x} dx \right)$$

let $y = (\vartheta - it)x \Rightarrow x = \frac{y}{(\vartheta - it)}$, and $dx = \frac{dy}{(\vartheta - it)}$, if $x = 0 \rightarrow y = 0$ and if $x = \infty \rightarrow y = \infty$

$$\phi_x(it) = \frac{1}{2} \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} \left(\vartheta \int_0^{\infty} \left(\frac{y}{(\vartheta - it)} \right)^{\beta v} e^{-y} \frac{dy}{(\vartheta - it)} + \vartheta^2 \int_0^{\infty} \left(\frac{y}{(\vartheta - it)} \right)^{\beta v + 1} e^{-y} \frac{dy}{(\vartheta - it)} \right. \\ \left. + 2\beta \int_0^{\infty} \left(\frac{y}{(\vartheta - it)} \right)^{\beta + \beta v - 1} e^{-y} \frac{dy}{(\vartheta - it)} + \vartheta\beta \int_0^{\infty} \left(\frac{y}{(\vartheta - it)} \right)^{\beta + \beta v} e^{-y} \frac{dy}{(\vartheta - it)} \right)$$

$$\phi_x(it) = \frac{1}{2} \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} \left(\frac{\vartheta}{(\vartheta + it)^{\beta v + 1}} \int_0^{\infty} y^{(\beta v + 1) - 1} e^{-y} dy + \frac{\vartheta^2}{(\vartheta + it)^{\beta v + 2}} \int_0^{\infty} y^{(\beta v + 2) - 1} e^{-y} dy \right. \\ \left. + \frac{2\beta}{(\vartheta + it)^{\beta + \beta v}} \int_0^{\infty} y^{(\beta + \beta v) - 1} e^{-y} dy + \frac{\vartheta\beta}{(\vartheta + it)^{\beta + \beta v + 1}} \int_0^{\infty} y^{(\beta + \beta v + 1) - 1} e^{-y} dy \right)$$

$$\phi_x(it) = \frac{1}{2} \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} \left(\frac{\vartheta \Gamma_{\beta v+1}}{(\vartheta + it)^{\beta v+1}} + \frac{\vartheta^2 \Gamma_{\beta v+2}}{(\vartheta + it)^{\beta v+2}} + \frac{2\beta \Gamma_{\beta+\beta v}}{(\vartheta + it)^{\beta+\beta v}} + \frac{\beta \vartheta \Gamma_{\beta+\beta v+1}}{(\vartheta + it)^{\beta+\beta v+1}} \right) \quad (23)$$

3.6 Moment Generating Function

It is a function that summarizes and defines the probability distribution, such as the mean and variance, as follows:

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx = \frac{1}{2} \int_0^{\infty} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta \beta x^{\beta}) e^{-(\vartheta x + x^{\beta}) + tx} dx$$

$$e^{-x^{\beta}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{\beta k}$$

$$M_x(t) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\vartheta \int_0^{\infty} x^{\beta k} e^{-(\vartheta-t)x} dx + \vartheta^2 \int_0^{\infty} x^{\beta k+1} e^{-(\vartheta-t)x} dx + 2\beta \int_0^{\infty} x^{\beta+\beta k-1} e^{-(\vartheta-t)x} dx + \vartheta \beta \int_0^{\infty} x^{\beta+\beta k} e^{-(\vartheta-t)x} dx \right)$$

let $y = \vartheta x \Rightarrow x = \frac{y}{\vartheta-t}$, and $dx = \frac{dy}{\vartheta-t}$, if $x = 0 \rightarrow y = 0$ and if $x = \infty \rightarrow y = \infty$

$$\begin{aligned} &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\vartheta \int_0^{\infty} \left(\frac{y}{\vartheta-t} \right)^{\beta k} e^{-y} \frac{dy}{\vartheta-t} + \vartheta^2 \int_0^{\infty} \left(\frac{y}{\vartheta-t} \right)^{\beta k+1} e^{-y} \frac{dy}{\vartheta-t} \right. \\ &\quad \left. + 2\beta \int_0^{\infty} \left(\frac{y}{\vartheta-t} \right)^{\beta+\beta k-1} e^{-y} \frac{dy}{\vartheta-t} + \vartheta \beta \int_0^{\infty} \left(\frac{y}{\vartheta-t} \right)^{\beta+\beta k} e^{-y} \frac{dy}{\vartheta-t} \right) \\ M_x(t) &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\vartheta}{(\vartheta-t)^{\beta k+1}} \int_0^{\infty} y^{(\beta k+1)-1} e^{-y} dy + \frac{\vartheta^2}{(\vartheta-t)^{\beta k+2}} \int_0^{\infty} y^{(\beta k+2)-1} e^{-y} dy \right. \\ &\quad \left. + \frac{2\beta}{(\vartheta-t)^{\beta+\beta k}} \int_0^{\infty} y^{(\beta+\beta k)-1} e^{-y} dy + \frac{\vartheta \beta}{(\vartheta-t)^{\beta+\beta k+1}} \int_0^{\infty} y^{(\beta+\beta k+1)-1} e^{-y} dy \right) \end{aligned}$$

$$M_x(t) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\vartheta \Gamma_{\beta k+1}}{(\vartheta-t)^{\beta k+1}} + \frac{\vartheta^2 \Gamma_{\beta k+2}}{(\vartheta-t)^{\beta k+2}} + \frac{2\beta \Gamma_{\beta+\beta k}}{(\vartheta-t)^{\beta+\beta k}} + \frac{\vartheta \beta \Gamma_{\beta+\beta k+1}}{(\vartheta-t)^{\beta+\beta k+1}} \right) \quad (24)$$

3.7 Factorial Moments Generating Function

This is a function used to extract the factorial moments of a random variable, as follows:

$$\mathcal{M}_x(t) = E(t^x) = \int_0^{\infty} t^x f(x) dx = \int_0^{\infty} e^{x \ln t} \frac{1}{2} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta \beta x^{\beta}) e^{-(\vartheta x + x^{\beta})} dx$$

$$\mathcal{M}_x(t) = \int_0^{\infty} \frac{1}{2} (\vartheta + \vartheta^2 x + 2\beta x^{\beta-1} + \vartheta \beta x^{\beta}) e^{-(\vartheta x + x^{\beta}) + x \ln t} dx$$

$$e^{-x^\beta} = \sum_{u=0}^{\infty} \frac{(-1)^u}{u!} x^{\beta u}$$

$$\mathcal{M}_x(t) = \sum_{u=0}^{\infty} \frac{(-1)^u}{u!} \left(\vartheta \int_0^{\infty} x^{\beta u} e^{-(\vartheta - lnt)x} dx + \vartheta^2 \int_0^{\infty} x^{\beta u+1} e^{-(\vartheta - lnt)x} dx \right. \\ \left. + 2\beta \int_0^{\infty} x^{\beta+\beta u-1} e^{-(\vartheta - lnt)x} dx + \vartheta\beta \int_0^{\infty} x^{\beta+\beta u} e^{-(\vartheta - lnt)x} dx \right)$$

$$\text{let } y = \vartheta x \Rightarrow x = \frac{y}{(\vartheta - lnt)}, \text{ and } dx = \frac{dy}{(\vartheta - lnt)}$$

If $x = 0 \rightarrow y = 0$ and if $x = \infty \rightarrow y = \infty$

$$\mathcal{M}_x(t) = \frac{1}{2} \sum_{u=0}^{\infty} \frac{(-1)^u}{u!} \left(\vartheta \int_0^{\infty} \left(\frac{y}{(\vartheta - lnt)} \right)^{\beta u} e^{-y} \frac{dy}{(\vartheta - lnt)} \right. \\ \left. + \vartheta^2 \int_0^{\infty} \left(\frac{y}{(\vartheta - lnt)} \right)^{\beta u+1} e^{-y} \frac{dy}{(\vartheta - lnt)} \right. \\ \left. + 2\beta \int_0^{\infty} \left(\frac{y}{(\vartheta - lnt)} \right)^{\beta+\beta u-1} e^{-y} \frac{dy}{(\vartheta - lnt)} \right. \\ \left. + \vartheta\beta \int_0^{\infty} \left(\frac{y}{(\vartheta - lnt)} \right)^{\beta+\beta u} e^{-y} \frac{dy}{(\vartheta - lnt)} \right)$$

$$\mathcal{M}_x(t) = \frac{1}{2} \sum_{u=0}^{\infty} \frac{(-1)^u}{u!} \left(\frac{\vartheta}{(\vartheta - lnt)^{\beta u+1}} \int_0^{\infty} y^{(\beta u+1)-1} e^{-y} dy \right. \\ \left. + \frac{\vartheta^2}{(\vartheta - lnt)^{\beta u+2}} \int_0^{\infty} y^{(\beta u+2)-1} e^{-y} dy + \frac{2\beta}{(\vartheta - lnt)^{\beta+\beta u}} \int_0^{\infty} y^{(\beta+\beta u)-1} e^{-y} dy \right. \\ \left. + \frac{\vartheta\beta}{(\vartheta - lnt)^{\beta+\beta u+1}} \int_0^{\infty} y^{(\beta+\beta u+1)-1} e^{-y} dy \right)$$

$$\mathcal{M}_x(t) = \frac{1}{2} \sum_{u=0}^{\infty} \frac{(-1)^u}{u!} \left(\frac{\vartheta \Gamma_{\beta u+1}}{(\vartheta - lnt)^{\beta u+1}} + \frac{\vartheta^2 \Gamma_{\beta u+2}}{(\vartheta - lnt)^{\beta u+2}} + \frac{2\beta \Gamma_{\beta+\beta u}}{(\vartheta - lnt)^{\beta+\beta u}} \right. \\ \left. + \frac{\vartheta\beta \Gamma_{\beta+\beta u+1}}{(\vartheta - lnt)^{\beta+\beta u+1}} \right) \quad (25)$$

4. Mean time to failure

Average failure time is a statistical measure used to estimate the lifespan of a particular system before it fails, and it is calculated as follows:

$$S_{(wxl)}(x) = \Pr(T > x)$$

$$MTTE = \int_0^{\infty} S_{(WXL)}(x) dx = \int_0^{\infty} \frac{1}{2} (2 + \vartheta x) e^{-(\vartheta x + x^\beta)} dx$$

$$\begin{aligned}
e^{-x^\beta} &= \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} x^{\beta a} \\
MTTE &= \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \left(\int_0^{\infty} x^{\beta a} e^{-(\vartheta x)} dx + \frac{1}{2} \int_0^{\infty} \vartheta x^{\beta a+1} e^{-(\vartheta x)} dx \right) \\
\text{let } y &= \vartheta x \Rightarrow x = \frac{y}{\vartheta}, \text{ and } dx = \frac{dy}{\vartheta} \\
\text{if } x &= 0 \rightarrow y = 0 \text{ and if } x = \infty \rightarrow y = \infty \\
MTTE &= \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \left(\frac{1}{\vartheta^{\beta a+1}} \int_0^{\infty} y^{\beta a} e^{-y} dy + \frac{1}{2\vartheta^{\beta a+1}} \int_0^{\infty} y^{\beta a+1} e^{-y} dy \right) \\
MTTE &= \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \left(\frac{1}{\vartheta^{\beta a+1}} \int_0^{\infty} y^{(\beta a+1)-1} e^{-y} dy + \frac{1}{2\vartheta^{\beta a+1}} \int_0^{\infty} y^{(\beta a+2)-1} e^{-y} dy \right) \\
MTTE &= \frac{1}{2} \sum_{a=0}^{\infty} \frac{(-1)^a}{a! \cdot \vartheta^{\beta a+1}} (2\Gamma_{\beta a+1} + \Gamma_{\beta a+2}) \tag{26}
\end{aligned}$$

5. Conclusion

A novel probability distribution was generated by combining the Whipple and x-Lindley distributions using a survival function multiplication methodology, resulting in a hybrid distribution called the Whipple x-Lindley. This construction method leverages the structural properties of both original distributions while enhancing the formal flexibility of the resulting model, particularly in representing tail behavior and varying failure rates. The statistical properties of the distribution were also explored, including moments around the origin and mean, variance, skewness and kurtosis coefficients, as well as tail behavior and function boundaries. These properties demonstrated that the proposed distribution exhibits a high degree of flexibility compared to traditional distributions, making it suitable for modeling data with high dispersion or skewness. Furthermore, graphical representations of the density, survival, and hazard functions were presented to illustrate the influence of these parameters on the overall shape of the distribution, highlighting the behavioral diversity the model can accommodate. The theoretical and graphical results support the potential of the Whipple x-Lindley distribution as an effective alternative model for survival analysis, reliability, engineering, and life science applications. Therefore, the proposed new distribution is not just a mathematical generalization but a valuable addition to the family of composite distributions.

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Conflict of Interest

This study presents no conflicts of interest.

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