



A New Study on Quasi 2-Normed Spaces: Geometric, Topological and Metrizable

Intisar Kareem Atia^{1*}  , and Salwa Salman Abed²  

^{1,2}Department of Mathematics, College of Education for Pure Sciences, Ibn Al-Haitham, University of Baghdad, Baghdad, Iraq

*Corresponding Author

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Abstract

This research paper addresses several aspects of the structure of quasi-2-normed spaces, which dates back to C. Park, as a broader class of quasi-modular spaces. It begins by reviewing the definition along with several diverse examples. Also, the derivation of a normed space, a quasi-normed space, and a quasi-2-metric space is presented from the current space and vice versa. The description of the ball in this space was presented as a non-convex, symmetric, balanced, and absorbing set, and this was proven. Topologically, it has been proven that the quasi-2-normed spaces are Hausdorff topological vector spaces upon the usual topology (a topology generated by a family of balls). Another topology by a separating family of a-quasi-norms has been defined, and its coincidence with the usual topological structure has been demonstrated. Throughout this study, the sufficient and necessary assumption for the normality of quasi-2-normed spaces, and the sufficient assumption for the metrizable property, have been proven.

Keywords: 2-normed space, Topological vector space, Separating, Semi-norms.

1. Introduction

Normed spaces are considered the core of the analysis, and their origins can be traced back to the 19th century in the fundamental work of pioneering researchers in this field¹. Always, normed spaces are metrizable; their metric topology generated by the norm is Hausdorff and more locally convex. These magnificent properties demonstrate the principal role of normed and Banach spaces in general analysis and applied mathematics². Some generalizations of normed spaces appeared, among which the 2-normed space takes an important place,³⁻¹². In 1960, a 2-norm function was introduced on ordered pairs of vectors, yielding a non-negative real number, replacing the concept of length with area^{13,14}. Generally, the 2-normed space is not metrizable, but it induces a family of semi-norms and hence a locally convex topology. Research continues to investigate convergence and fixed-point theory in this framework and its extensions, setting- n -normed^{15,16}. The weakness of the classical standard triangle inequality leads to the emergence of a quasi-standard space through many developments by mathematicians. Although every quasi-norm induces a topology, this topology is not locally convex in general¹⁷⁻¹⁹. Hamel's basis²⁰ provides a natural algebraic framework for describing the linear structure of such spaces. Therefore, some basic facts may be incorrect. This led to differences between normed and quasi-normed spaces²¹⁻²³. This difference parallels the distinction between metric spaces and generalized distance structures, including 2-metric²⁴ and b2-metric spaces²⁵. Studies have also emerged relating to 2-inner product spaces (2-pre-Hilbert spaces), and have dealt in particular with orthogonal sets and standard orthogonal sets in these spaces²⁵⁻³⁰. In this paper, a diverse study of several aspects of quasi-2-normed spaces will be presented

2. Preliminaries

Recall the following definition

2.1 Definition: Let Ω be a real vector space with $\dim \Omega \geq 2$. The function $q_s: \Omega \times \Omega \rightarrow [0, \infty)$ such that the following hold:

- $q_s(s, d) \geq 0$ for all $s, d \in \Omega$ and $q_s(s, d) = 0$ if and only if the set $\{s, d\}$ is linearly dependent,
- $q_s(s, d) = q_s(d, s)$, for all $s, d \in \Omega$,
- $q_s(s, \alpha d) = |\alpha| q_s(s, d)$ for all $s, d \in \Omega, \alpha \in R$,
- $q_s(s, d + z) \leq k(q_s(s, d) + q_s(d, z))$ for all $s, d \in \Omega$ and constant $k \geq 1$.

Then the ordered pair (Ω, q_s) is named a quasi-2-normed space and is denoted by q2-normed space¹⁷.

2.2. Proposition: Every 2-normed space is a q2-normed space when the constant $k = 1$ ²¹.

The converse is not true. The next examples illustrate this fact.

2.3. Example: Let $\Omega = R^3$ and $s = s_1i + s_2j + s_3k$, $d = d_1i + d_2j + d_3k \in R^3$. Define $q_s(s, d) = h|s_{i_0}d_{i_0+1} - s_{i_0+1}d_{i_0}| + \sum_{i \neq i_0}^3 |s_i d_{i+1} - s_{i+1} d_i|$ where $|s_{i_0}d_{i_0+1} - s_{i_0+1}d_{i_0}| = \min \{|s_i d_{i+1} - s_{i+1} d_i| : 1 \leq i \leq 3\}$ such that $s_4 = s_1$, $d_4 = d_1$ and $h > 1$ then (R^3, q_s) is a q2-normed space, but is not 2-normed space for all $h > 1$ ¹⁸.

2.4. Example: Let $P_2[0,1]$ be the set of real polynomials of degree 2. By regarding usual addition and scalar multiplication, $P_2[0,1]$ is a linear vector space. Let s_1, s_2, s_3 and s_4 be distinct fixed points in $[0,1]$. Define the q2- norm on $P_2[0,1]$ as

$$q_s(f, g) = k|f(s_{i_0})g'(s_{i_0}) - f'(s_{i_0})g(s_{i_0})| + \sum_{i \neq i_0}^4 |f(s_i)g'(s_i) - f'(s_i)g(s_i)|,$$

where $|f(s_{i_0})g'(s_{i_0}) - f'(s_{i_0})g(s_{i_0})| = \min \{|f(s_i)g'(s_i) - f'(s_i)g(s_i)| : 1 \leq i \leq 4\}$ and $k > 1$. Then (P_2, q_s) is a q2-normed space and not a 2-normed space¹⁸.

Another new example is:

2.5. Example: Consider $\Omega = R^2, s = (s_1, s_2), w = (w_1, w_2)$ and $\det(s, w) = \det \begin{pmatrix} s_1 & s_2 \\ w_1 & w_2 \end{pmatrix} = s_1 w_2 - s_2 w_1$, define $q_s(s, w) = |\det(s, w)| + \sqrt{|\det(s, w)|}$ for all $s, w \in \Omega$ is not q2-normed space.

Proof: Apply the conditions of Definition (2.1)

- Clearly $q_s(s, w) \geq 0$ if and only if s, w linearly dependent.
- $q_s(s, w) = |\det(s, w)| + \sqrt{|\det(s, w)|} = |\det(w, s)| + \sqrt{|\det(w, s)|} = q_s(w, s)$
- $q_s(s, \alpha w) = |\det(s, \alpha w)| + \sqrt{|\det(s, \alpha w)|} = |\alpha| |\det(s, w)| + \sqrt{|\alpha| |\det(s, w)|}$, but $|\alpha| q_s(s, w) = |\alpha| |\det(s, w)| + |\alpha| \sqrt{|\det(s, w)|}$. This is not equal $\sqrt{|\alpha|} \neq |\alpha|$

(Ω, q_s) is not q2-normed.

There are some inducing q2-normed spaces from 2-normed spaces or normed spaces from 2-normed spaces, etc., as follows

2.6. Proposition: If $(\Omega, \|\cdot\|)$ is a 2-normed space then (Ω, q_s) is a q2-normed space when $q_s(s, d) = 2 \|s, d\|$, for all $s, d \in \Omega$ ¹⁹.

2.7. Proposition: If $(\Omega, \|\cdot\|)$ is a 2-normed space and $w \geq \frac{1}{2}$, then (Ω, q_s) is a q2-normed space if $q_s(s, d) = z \|s, d\| + w \|s, d\|$, for all $s, d \in \Omega$ ¹⁹.

It is appropriate to give cases for a 2-norm (or a q2-norm) derived from a norm that is induced by an inner product.

2.8. Proposition: If $(\Omega, \|\cdot\|)$ is a normed space with $\dim \Omega \geq 2$ and $\|\cdot\|$ induced by an inner product, then (Ω, q_s) is a 2-normed space (and then a q2-normed space),

where $q_s(s, d) = \|s\| \|d\| \sin \theta_{(s,d)}$, and $\theta_{(s,d)} := \text{Angle of vectors } s \text{ and } d$.

Proof: for any s, d, z in $\Omega, \alpha \in R$

- clearly $q_s(s, d) \geq 0$ if and only if s, d are linearly dependent,
- $q_s(s, d) = \|s\| \|d\| \sin \theta_{(s,d)} = \|d\| \|s\| \sin \theta_{(d,s)} = q_s(d, s)$,

- $q_s(s, \alpha d) = \|s\| \|\alpha d\| \sin \theta_{(s, \alpha d)}$
 $= |\alpha| \|s\| \|d\| \sin \theta_{(s, d)} = |\alpha| q_s(s, d),$
- $q_s(s, d + z) = \|s\| \|d + z\| \sin \theta_{(s, d+z)}$
 $\leq \|s\| \|d\| \sin \theta_{(s, d)} + \|s\| \|z\| \sin \theta_{(s, z)}$
 $\leq q_s(s, d) + q_s(s, z).$

The author in ¹⁷ defined a $(2, p)$ -norm (and $(2, p)$ -normed space) from a q2-normed (q2-normed space).

2.9. Proposition: If (Ω, q_s) is a q2-normed space, then there exists $0 < p \leq 1$ and an equivalent q2-normed (denoted $\bar{q}_s$) on Ω such that $[\bar{q}_s(s + d, w)]^p \leq [\bar{q}_s(s, w)]^p + [\bar{q}_s(d, w)]^p$ for all $s, d, w \in \Omega$ ¹⁷.

Also in ¹⁷, the researcher defined a quasi-norm from a q2-normed space.

2.10. Proposition: If (Ω, q_s) is a q2-normed space, $p \geq 1$ and $\{z^*, w^*\}$ is a linearly independent subset of Ω . Then

$\|d\|_1 = [(q_s(d, z^*))^p + (q_s(d, w^*))^p]^{\frac{1}{p}}, d \in \Omega$ and $\|d\|_2 = \max \{q_s(d, z^*), q_s(d, w^*)\}, d \in \Omega$ are two quasi-norms on Ω ¹⁷.

Recalling some needed definitions and facts for this work.

2.11. Lemma: Let Ω be a vector space with $\dim \geq 2$. Then for any nonzero $s \in \Omega, \exists d \in \Omega$ such that $\{s, d\}$ is a linearly independent set².

2.12. Lemma: Any vector space has a basis (specifically, a Hamel basis²⁰).

2.13. Definition: Let Ω be a vector space over the field F and T be a topology on Ω . The pair (Ω, T) is called a topological vector space (TVS) if the operations $+, \cdot$ on Ω are continuous up on T ².

2.14. Definition: Let (Ω, q_s) be a TVS is called locally convex if there is a basis of neighborhoods of the origin in Ω composed of convex sets².

2.15. Proposition: For any TVS Ω , the following are equivalent

- Ω is Hausdorff (T_2).
- The intersection of all neighborhoods of the origin 0 is just $\{0\}$.
- Ω is T_0 ².

2.16. Definition: A TVS (Ω, T) is called metrizable if its topology T induced by a metric (that is, compatible with the vector space Ω)¹.

2.17. Theorem: A TVS has a countable local base if and only if it is metrizable³.

2.18. Definition: A TVS (Ω, T) is called normable if its topology T induced by norm $\|\cdot\|$ ¹.

2.19. Theorem: A TVS Ω is normable if there is a convex, bounded open set containing the origin².

Here, we will present a generalization of 2-metric space³⁰, that is named q2-metric space (quasi-2-metric)

2.20. Definition: Let Ω be a non-empty set and $\Delta: \Omega^3 \rightarrow R^+$ be a map satisfying the following properties:

- For any two elements s, d in Ω , there exists $z \in \Omega$ such that $\Delta(s, d, z) \neq 0$,
- $\Delta(s, d, z) = 0$ if at least two of the three points are the same,
- $\Delta(s, d, z) = \Delta(s, z, d) = \Delta(d, s, z) = \Delta(d, z, s) = \Delta(z, s, d) = \Delta(z, d, s)$ for all $s, d, z \in \Omega$,
- $\Delta(s, d, z) \leq k(\Delta(s, d, w) + \Delta(d, z, w) + \Delta(z, s, w))$ for all $s, d, z, w \in \Omega$ and $k \geq 1$.

Then Δ is called the q2-metric on Ω and (Ω, Δ) is called a q2-metric space.

2.21. Example: Let (Ω, q_s) be a q2-normed space with $\dim \geq 2, q_s(s, d) = 2 \|s, d\|$, for all $s, d \in \Omega$. Define function $\Delta: \Omega \times \Omega \times \Omega \rightarrow R^+$ by

$\Delta(s, d, z) = 2 \|s - z, d - z\|$ for all $s, d, z \in \Omega$

- Clearly $\Delta(s, d, z) = 2 \|s - z, d - z\| \neq 0$
- $\Delta(s, d, z) = 0 \Leftrightarrow 2 \|s - z, d - z\| = 0$ if and only if $(s - z)$ and $(d - z)$ are linearly dependent, then $s - z = \alpha (d - z), \alpha \in R$

$= 2 \| \alpha (d - z), d - z \| = 2 | \alpha | \| d - z, d - z \| = \Delta(d, d, z)$, these are the two elements that are the same.

- $\Delta(s, d, z) = 2 \| s - z, d - z \| = 2 \| d - z, s - z \| = \Delta(d, s, z)$

In the same way, one can prove it.

- $\Delta(s, d, z) = 2 \| s - z, d - z \| = 2 \| (s - t) + (t - z), (d - t) + (t - z) \|$
 $\leq b(2 \| (s - t) + (t - z), (d - t) \| + 2 \| (s - t) + (t - z), (t - z) \|)$
 $\leq b(b(2 \| d - t, s - t \| + 2 \| d - t, t - z \| + 2 \| t - z, s - t \| + 2 \| t - z, t - z \|))$
 $\leq b^2(2 \| s - t, d - t \| + 2 \| d - t, z - t \| + 2 \| s - t, z - t \|)$

Put $k = b^2$

$\leq k(\Delta(s, d, t) + \Delta(d, z, t) + \Delta(s, z, t))$

Then (Ω, q_s) is a q2-metric space.

2.22. Definition: Let (Ω, q_s) be a q2-normed space and $\{s_n\}$ is a sequence in Ω is called a convergent sequence if there is an $s \in \Omega$ such that $\lim_{n \rightarrow \infty} q_s(s_n - s, z) = 0$ for every z in Ω ¹⁹.

2.23. Definition: Let (Ω, q_s) be a q2-normed space and $\{s_n\}$ is a sequence in Ω is called a Cauchy sequence if and only if $\lim_{n, m \rightarrow \infty} q_s(s_n - s_m, z) = 0$, for every z in Ω ¹⁹.

2.24. Definition: A q2-normed space Ω is called complete if every Cauchy sequence in Ω is convergent to s in Ω ¹⁹.

3. Main Results

Let Ω be a q2-normed space. The ball in (Ω, q_s) defined as, $B(s, r) = \{d \in \Omega ; q_s(d - s, w) < r\}$, for $w \in \Omega$. The following Theorem gives some geometric properties of $B(s, r)$.

3.1. Theorem: Any ball $B(s, r)$ in a q2-normed space is not a convex set, but it is symmetric, balanced, and absorbing.

Proof: Convexity: For any $b, d \in B(s, r)$ and $t \in [0, 1]$

$$q_s(tb + (1 - t)d, w) \leq k(q_s(tb, w) + q_s((1 - t)d, w))$$

$$\leq k|t|q_s(b, w) + k|1 - t|q_s(d, w) \leq kr$$

But, kr may be greater than r , so $tb + (1 - t)d$ may not be in $B(s, r)$. Hence, it is not convex.

Symmetry: It is satisfied, since for any $s \in \Omega$, $q_s(-s, w) = |-1|q_s(s, w)$

implies that $B(s, r) = -B(s, r)$

This shows that $B(s, r)$ is symmetric.

Balance: For all α with $|\alpha| \leq 1$ and $d \in B(s, r)$, $q_s(\alpha d, w) = |\alpha|q_s(d, w) \leq q_s(d, w) < r$

That is, $\alpha d \in B(s, r)$ for all $d \in B(s, r)$. Hence $B(s, r)$ is balanced.

Absorbing: Let $d \in \Omega$. If $\{d, w\}$ is linearly dependent then $q_s(d, w) = 0 < 1$ and so, $d \in B(s, r) = tB(s, r)$ where $t = 1$.

On the other hand, if we take $t = \frac{2q_s(d, w)}{r} > 0$, then $q_s\left(\frac{1}{t}d, w\right) = \frac{1}{t}q_s(d, w) = \frac{r}{2} < r$.

This shows that $d \in tB(s, r)$ for some $t > 0$ hence $B(s, r)$ is absorbing.

Note that the following

3.2. Theorem: A q2-normed space is a q2-metric space.

Proof: Define the following function $\Delta: \Omega \times \Omega \times \Omega \rightarrow R^+$ by

$\Delta(s, d, z) = q_s(s - z, d - z)$ for all $s, d, z \in \Omega$

- Clearly $\Delta(s, d, z) = q_s(s - z, d - z) \neq 0$

- $\Delta(s, d, z) = 0 \Leftrightarrow q_s(s - z, d - z) = 0$ if and only if $(s - z)$ and $(d - z)$ are linearly dependent, then $s - z = \alpha(d - z)$, $\alpha \in R$

$$q_s(s - z, d - z) = q_s(\alpha(d - z), d - z) = |\alpha|q_s(d - z, d - z) = \Delta(d, d, z),$$

These are the two elements that are the same.

- $\Delta(s, d, z) = q_s(s - z, d - z) = q_s(d - z, s - z) = \Delta(d, s, z)$

In the same way, one can prove it.

- $\Delta(s, d, z) = q_s(s - z, d - z) = q_s((s - t) + (t - z), (d - t) + (t - z))$
 $\leq b(q_s(s - t) + (t - z), (d - t) + q_s(s - t) + (t - z), (t - z))$

$$\begin{aligned} &\leq b (b(q_s(d-t, s-t) + q_s(d-t, t-z) + b(q_s(t-z, s-t) + (t-z, t-z))) \\ &\leq b^2(q_s(s-t, d-t) + q_s(d-t, z-t) + q_s(s-t, z-t)) \\ &\text{Put } k = b^2 \end{aligned}$$

$$\leq k (\Delta(s, d, t) + \Delta(d, z, t) + \Delta(s, z, t)).$$

In what follows, some topological aspects of q2-normed spaces are discussed.

The collection $\mathfrak{B} = \{B(s, r): s \in \Omega, r > 0\}$ generates a topology on Ω , say, $\tau_{\mathfrak{B}}$. For the needed basic topological preliminaries, see².

3.3. Theorem: The topological space $(\Omega, \tau_{\mathfrak{B}})$ is T_0 .

Proof: Let $d, s \in \Omega$ and $d \neq s$, $d - s$ is a non-zero vector.

Since $\dim \Omega \geq 2$, $w \in \Omega$ such that $d - s$ and w are linearly independent by Lemma (2.11).

$$\text{Put } \delta = \frac{1}{2} q_s(d - s, w) > 0$$

Then, it is clear that $s \in B(s, \delta)$ but $d \notin B(s, \delta)$.

This from $q_s(d - s, w) = 2\delta > \delta$. Therefore, $\tau_{\mathfrak{B}}$ is T_0 -topological space.

3.4. Theorem: Every q2-normed space is a topological vector space (TVS).

Proof: It is required to prove that the addition and multiplication operations are continuous with respect to $\tau_{\mathfrak{B}}$ see². Let $w \in \Omega$, $\varepsilon > 0$, $\delta > 0$ such that $\delta = \frac{\varepsilon}{2k}$

If $q_s(s - s_0, w) < \delta$ and $q_s(d - d_0, w) < \delta$

$$\begin{aligned} q_s(s + d - (s_0 + d_0), w) &= q_s((s - s_0) + (d - d_0), w) \\ &\leq k q_s((s - s_0, w) + q_s(d - d_0, w)) \\ &\leq k \left(\frac{\varepsilon}{2k} + \frac{\varepsilon}{2k} \right) \leq \varepsilon \end{aligned}$$

Also, $q_s(\alpha s - \alpha_0 s_0, w) = q_s(\alpha(s - s_0) + (\alpha - \alpha_0)s_0, w)$

$$\leq k(|\alpha| q_s(s - s_0, w) + |\alpha - \alpha_0| q_s(s_0, w))$$

$$\leq k(|\alpha| \delta + |\alpha - \alpha_0| q_s(s_0, w))$$

$$\text{Let } \delta = \frac{\varepsilon}{k(|\alpha| + |\alpha - \alpha_0| q_s(s_0, w))}$$

$$q_s(\alpha s - \alpha_0 s_0, w) \leq \varepsilon$$

Then a q2-normed space Ω is TVS.

From the above, a q2-normed space satisfies the following important fact:

3.5. Remark:

- By Theorems (3.3 and 3.4) and Proposition (2.15), the q2-normed space is T_2 .
- A q2-normed space is not a locally convex TVS, because the ball is not convex.

The purpose of the next step is to define a topology by a separating family of q-seminorms on Ω starting with the following:

3.6. Definition: Let ρ be a function defined on a real vector space Ω , if ρ satisfies the following:

- $\rho(s + d) \leq k(\rho(s) + \rho(d))$, for any $s, d \in \Omega$, $k \geq 1$
- $\rho(\alpha s) = |\alpha| \rho(s)$, for any $s \in \Omega$ and $\alpha \in R$

Then ρ is named a q-seminorm.

3.7. Definition: A family of q-semi-norms $\wp = \{\rho_i: i \in I\}$ is named a separating family if $\forall s \in \Omega, \exists \rho_i \in \wp$ such that $\rho_i(s) \neq 0$.

3.8. Definition: Let Ω be a vector space and $\wp = \{\rho_i: i \in I\}$ be a family of q-seminorms on Ω . Then $B_r^i(s) = \{d \in \Omega: \rho_i(s - d) < r\}$ is the i th ball of radius $r > 0$ and centered at $s \in \Omega$.

3.9. Definition: Let Ω be a vector space and $\wp = \{\rho_i: i \in I\}$ be a family of q-seminorms then $\mathfrak{F} = \{B_r^i(s): s \in \Omega, r > 0, i \in I\}$ is the collection of all balls in Ω . The topology $\tau_{\mathfrak{F}}$ generated by $\mathfrak{F}$ is named the topology induced by the family of q-seminorms $\wp$ and denoted by $\tau_{\wp}$.

3.10. Remark: Let B be the collection of all finite intersections of the elements of $\mathfrak{F}$ forms a base for the topology $\tau_{\wp}$.

3.11. Proposition: In a q2-normed space, vector addition and scalar multiplication are continuous for the topology $\tau_{\wp}$.

Proof: For any $s, d \in \Omega$, $\varepsilon > 0$ and $\rho_i \in \wp$ such that $\rho_i(s - s_0) < \frac{\varepsilon}{2k}$ and $\rho_i(d - d_0) < \frac{\varepsilon}{2k}$; then
 $\rho_i(s + d - (s_0 + d_0)) \leq k(\rho_i(s - s_0) + \rho_i(d - d_0)) \leq k(\frac{\varepsilon}{2k} + \frac{\varepsilon}{2k}) \leq \varepsilon$
 Also, for any $\lambda, M \in R \exists \delta_1, \delta_2 > 0$ such that $M - \lambda < \delta_1, \rho_i(s - d) < \delta_2$
 $\delta_1 < \frac{\varepsilon}{2k\rho_i(d)}, \delta_2 < \frac{\varepsilon}{2k}(|\lambda| + \frac{\varepsilon}{2k\rho_i(d)})^{-1}$
 $\rho_i(Ms - \lambda d) = \rho_i((M - \lambda)s + \lambda(s - d))$
 $\leq k(|M - \lambda|\rho_i(s) + |\lambda|\rho_i(s - d))$
 $\leq k(\delta_1\rho_i(d) + \rho_i(s - d) + |\lambda|\delta_2)$
 $\leq k(\delta_1\rho_i(d) + \delta_2(\delta_1 + |\lambda|))$
 $\leq k(\frac{\varepsilon}{2k\rho_i(d)} \cdot \rho_i(d) + \frac{\varepsilon}{2k}(|\lambda| + \frac{\varepsilon}{2k\rho_i(d)})^{-1}(\frac{\varepsilon}{2k\rho_i(d)} + |\lambda|)) \leq \varepsilon.$

3.12. Proposition: $\wp$ is a separating family of q-seminorms if and only if $\tau_\wp$ is Hausdorff (T_2).

Proof: For any $s, d \in \Omega$, $s \neq d$ implies that $s - d \neq 0$, take $\varepsilon = \frac{\rho_i(s-d)}{2k} > 0$ for any $\rho_i \in \wp$.

The neighborhoods s and d , respectively are $U_s = s + U(\rho_i, \varepsilon)$, $U_d = d + U(\rho_i, \varepsilon)$

To prove that $U_s \cap U_d = \emptyset$. Assume the contrary $a \in U_s \cap U_d$ and

$$\rho_i(a - s) < \varepsilon, \rho_i(a - d) < \varepsilon$$

$$\rho_i(s - d) = \rho_i(s - a + a - d) \leq k(\rho_i(s - a) + \rho_i(a - d))$$

$$\rho_i(s - d) < k(\frac{\rho_i(s - d)}{2k} + \frac{\rho_i(s - d)}{2k}) < \rho_i(s - d)$$

This is a contradiction, then $\tau_\wp$ is Hausdorff.

Conversely, let (Ω, q_s) be a Hausdorff space suppose that $0 \neq s \in \Omega$, $\rho_i(s) = 0$

for any $s, d \in \Omega$, $s \neq d$

There exists a neighborhood of s is $U_s = \{z \in \Omega, \rho_i(z - s) < \varepsilon\}$ and a neighborhood of d is $U_d = \{z \in \Omega, \rho_i(z - d) < \varepsilon\}$

$$\rho_i(z - s) = \rho_i(z - d) = 0 < \varepsilon$$

Hence, $z \in U_s \cap U_d \Rightarrow U_s \cap U_d \neq \emptyset$. That leads to a contradiction. Then $\rho_i(s) \neq 0$.

3.13. Remark: $\tau_\wp$ is a topological vector space if and only if $\wp$ is a separating family of q-seminorms by Propositions (3.11 and 3.12).

3.14. Theorem: A q2-normed space (Ω, q_s) and B be a Hamel basis of Ω Then the family $\wp = \{\rho_i: i \in I\}$ constructs a separating family of q-seminorms on Ω .

Proof: For any $a \in B$, show the following

- For any $s, d \in \Omega$, $k \geq 1$
 $\rho_a(s + d) = q_s(s + d, a)$
 $\leq k(q_s(s, a) + q_s(d, a))$
 $\leq k(\rho_a(s) + \rho_a(d))$

- For any $s \in \Omega$ and $\alpha \in R$

$$\rho_a(\alpha s) = q_s(\alpha s, a) = |\alpha| q_s(s, a) = |\alpha| \rho_a(s)$$

So, ρ_a is a q-seminorm for any $a \in B$ and $\wp = \{\rho_a: a \in B\}$ constructs a family of q-seminorms on Ω . Finally, for $s \in \Omega, \exists d \in B$ such that $\{s, d\}$ is linearly independent. Then, $q_s(s, d) \neq 0$ hence $\rho_d(s) \neq 0$. Therefore, $\wp$ constructs a separating family of q-seminorms.

The next theorem leads to the conclusion that a q2-normed space is a TVS by $\tau_\wp$ (where $\tau_\wp$ is a topology by separating family of q-seminorms).

3.15. Theorem: Let (Ω, q_s) be a q2-normed space, then the topology of $(\Omega, \tau_\wp)$ coincides with the topological vector space $(\Omega, \tau_\wp)$.

Proof: Suppose that Δ is q2-metric induced by q2-norm. For any $a, b \in \Omega$ and $\varepsilon > 0$. Then the neighborhood of a and b is

$$U_\varepsilon(a, b) = \{s \in \Omega: \Delta(a, b, s) < \varepsilon\} = \{s \in \Omega: \Delta(b, s, a) < \varepsilon\}$$

$$= \{s \in \Omega: q_s(b - a, s - a) < \varepsilon\}$$

$$= \{s \in \Omega: q_s(a-s, b-a) < \varepsilon\}$$

$\wp = \{\rho_i: i \in I\}$ is the separating family of q-semi norms where H is a Hamel basis of Ω . In the first step, it must prove that $g = \{B_r^a(s): s \in \Omega, r > 0, a \in H\}$ is a subset of $\mathcal{U} = \{U_\varepsilon(a, b): a, b \in \Omega, \varepsilon > 0\}$ consider any element $B_r^a(s)$ from g .

$$\begin{aligned} B_r^a(s) &= \{d \in \Omega: p_a(s-d) < r\} \\ &= \{d \in \Omega: q_s(s-d, a) < r\} \\ &= \{d \in \Omega: q_s(s-d, a+s-s) < r\} \\ &= U_r(s, a+s) \end{aligned}$$

Then $B_r^a(s) \in \mathcal{U}$. So, $g \subseteq \mathcal{U}$. And $\tau_\wp$ is a subset of the natural topology $\tau_{\mathfrak{B}}$ of the q2-normed space Ω .

Conversely, suppose that $U_\varepsilon(a, b) \in \mathcal{U}$ and $z \in U_\varepsilon(a, b)$ then obtaining $q_s(a-s, b-a) < \varepsilon$, put $q_s(a-s, b-a) = \hat{\varepsilon}$, so $\varepsilon - \hat{\varepsilon} > 0$.

Now, there exists $z_1, z_2, \dots, z_n \in \mathcal{B}$ and $c_1, c_2, \dots, c_n \in \mathbb{R}$ such that $b-a = c_1 z_1 + c_2 z_2 + \dots + c_n z_n$; put $\delta = \frac{\frac{\varepsilon}{k} - \hat{\varepsilon}}{|c_1| + \dots + |c_n|}$

Then, for any $d \in \cap_{i=1}^n B_\delta^{a_i}(s)$, we obtain $p_{a_i}(s-d, a_i) < \delta$ for all $i = 1, 2, \dots, n$
 $q_s(s-d, a_i) < \delta$ for all $i = 1, 2, \dots, n$

$$\begin{aligned} q_s(s-d, b-a) &\leq \sum_{i=1}^n |c_i| \cdot q_s(s-d, a_i) < \delta \cdot \sum_{i=1}^n |c_i|. \\ q_s(s-d, b-a) &< \frac{\varepsilon}{k} - \hat{\varepsilon} \end{aligned}$$

This implies that $q_s(a-d, b-a) \leq k(q_s(a-s, b-a) + q_s(s-d, b-a)) \leq k(\hat{\varepsilon} + \frac{\varepsilon}{k} - \hat{\varepsilon}) \leq \varepsilon$

Then, $d \in U_\varepsilon(a, b)$ and so $s \in \cap_{i=1}^n B_\delta^{a_i}(s) \subseteq U_\varepsilon(a, b)$

Therefore, $U_\varepsilon(a, b)$ is an open set in $(\Omega, \tau_\wp)$. It means that $\mathcal{U} \subseteq \tau_\wp$. Thus, $\tau_\wp$ coincides with $\tau_{\mathfrak{B}}$ on Ω .

3.16. Remark: Note that the set of all possible finite intersections of the elements $B = \{\cap_{i=1}^m B_{r_i}^{a_i}(s_i): s_i \in \Omega, a_i \in \mathcal{B}, r_i > 0, m \in \mathbb{N}\}$ is a base for the natural topology of $(\Omega, \tau_{\mathfrak{B}})$.

3.17. Proposition:

For $a \in \mathcal{B}$ and $n \in \mathbb{N}$ defines $w(a, n) = B_{\frac{1}{n}}^a(0)$ then $B_0 = \{\cap_{i=1}^m w(a_i, n_i); a_i \in \mathcal{B}, n_i \in \mathbb{N}, m \in \mathbb{N}\}$ is a local base at 0.

Proof: Let $\mathcal{E}$ be an open set containing 0. Then, there exists $\cap_{i=1}^m B_{r_i}^{a_i}(s_i) \in \mathcal{B}$ such that $0 \in \cap_{i=1}^m B_{r_i}^{a_i}(s_i) \subseteq \mathcal{E}$.

Now for $i = 1, 2, \dots, m$, it follows that $0 \in B_{r_i}^{a_i}(s_i)$ and so $\dot{r}_i = q_s(s_i, a_i) < \frac{r_i}{k}$

Choose $n_i \in \mathbb{N}$ such that $\frac{1}{n_i} < \frac{r_i}{k} - \dot{r}_i$.

Then, $d \in V(a_i, n_i) \Rightarrow d \in B_{\frac{1}{n_i}}^{a_i}(0) \Rightarrow q_s(d, a_i) < \frac{1}{n_i}$.

$$q_s(d-s_i, a_i) \leq k(q_s(d, a_i) + q_s(s_i, a_i)) \leq k \left(\frac{1}{n_i} + \dot{r}_i \right) \leq k \cdot \frac{r_i}{k} \leq r_i$$

Hence, $0 \in \cap_{i=1}^m V(a_i, n_i) \subseteq \cap_{i=1}^m B_{r_i}^{a_i}(s_i) \subseteq \mathcal{E}$. Then B_0 is a local base at 0.

3.18. Theorem: Let (Ω, q_s) be a q2-normed space with a countable Hamel basis, then the topology induced by the q2-normed space is metrizable.

Proof: Let (Ω, q_s) be a q2-normed space with a countable Hamel basis, then $B_0 = \{\cap_{i=1}^m w(a_i, n_i); a_i \in \mathcal{B}, n_i \in \mathbb{N}, m \in \mathbb{N}\}$ is countable.

Therefore, the topology induced by the q2-normed space on Ω has a countable local base. So, is metrizable by Theorem (2.17).

3.19. Definition: A q_2 -normed space (Ω, q_s) is satisfy property P_j for some $j \in N$ if linearly independent elements $a_1, a_2, \dots, a_j$ in Ω exists for every sequence of elements $s, s_1, s_2, \dots$ of Ω , $\lim_{n \rightarrow \infty} q_s(s - s_n, a_i) = 0$ for all $i = 1, 2, \dots$, implies $\lim_{n \rightarrow \infty} q_s(s - s_n, a) = 0$ for each $a \in \Omega$, i.e. $s_n \rightarrow s$ in Ω .

3.20. Remark: Let (Ω, q_s) be a finite-dimensional q_2 -normed space with $\dim j \geq 2$ then Ω satisfies the property P_j .

Proof: Clearly.

3.21. Theorem: Let (Ω, q_s) be a q_2 -normed space and satisfy the property P_j for some $j \in N$, then it is normable.

Proof: By hypothesis, there exist linearly independent elements $a_1, a_2, \dots, a_j \in \Omega$, such that for every sequence $s, s_1, s_2, s_3, \dots$ in Ω ,

$\lim_{n \rightarrow \infty} q_s(s - s_n, a_i) = 0$ for all $i = 1, 2, \dots, j$, implies that $\lim_{n \rightarrow \infty} q_s(s - s_n, a) = 0$ for all $a \in \Omega$.

Also, a Hamel basis $\mathcal{B}$ of Ω exists such that $\{a_1, a_2, \dots, a_j\} \subset \mathcal{B}$

From Theorems (3.14) and (3.15), we get that the topology of (Ω, q_s) coincides with TVS.

Assume that, $\forall a \in \mathcal{B}$ and $\varepsilon > 0, \exists \delta > 0$ such that $\bigcap_{i=1}^j B_\delta^{a_i}(0) \subseteq B_\varepsilon^a(0)$

If not, then $\exists a_0 \in \mathcal{B}$ and $\varepsilon_0 > 0$ for which no such δ exists. Thus, $\forall \delta = \frac{1}{k}$ where $k \in N$, there exists $s_k \in \bigcap_{i=1}^j B_\delta^{a_i}(0) \setminus B_{\varepsilon_0}^{a_0}(0)$. So, $\forall n \in N, q_s(s_n, a_0) \geq \varepsilon_0$.

On the other side,

$\lim_{n \rightarrow \infty} q_s(s_n, a_i) = 0$ for all $i = 1, 2, \dots, j$. This is a contradiction to the supposition.

Now, $B_\delta^{a_i}(0) = \delta B_1^{a_i}(0) = \delta(\bigcap_{i=1}^j B_1^{a_i}(0) \subseteq B_\varepsilon^a(0))$ and $\bigcap_{i=1}^j B_1^{a_i}(0) \subseteq \frac{1}{\delta} B_\varepsilon^a(0)$.

Put $c = \frac{1}{\delta}$. Then, for each $t \geq c$, obtaining $\bigcap_{i=1}^j B_1^{a_i}(0) \subseteq t B_\varepsilon^a(0)$

Now, it is proven that $B = \bigcap_{i=1}^j B_1^{a_i}(0)$ is bounded.

Consider any open set G containing 0. $\bigcap_{i=1}^k V(b_i, n_i) \subseteq G$ for some $b_i \in \mathcal{B}, n_i, k \in N$, there exists $\varepsilon > 0$ such that $\varepsilon < \frac{1}{n_i}$ for each $i = 1, 2, \dots, k$. So $\bigcap_{i=1}^k B_\varepsilon^{b_i}(0) \subseteq \bigcap_{i=1}^k V(b_i, n_i) \subseteq G$. There

exists $s_i > 0$ such that $\bigcap_{i=1}^j B_1^{a_i}(0) \subseteq t B_\varepsilon^{b_i}(0), \forall t \geq c_i$. Take $c = \sup\{c_1, c_2, \dots, c_k\}$. Then, for all $t \geq c$ we have $B = \bigcap_{i=1}^j B_1^{a_i}(0) \subseteq t(\bigcap_{i=1}^k B_\varepsilon^{b_i}(0)) \subseteq tG$, for all $t \geq c$.

Hence B is a bounded open set containing 0. On the other side, B is the intersection of some balls, the topology of (Ω, q_s) is normable.

The converse of this theorem is also true.

3.22. Theorem: If (Ω, q_s) is a q_2 -normed space and normable then it satisfies the property (P_j) for some $j \in N$.

Proof: Let G be a bounded open set containing 0 such that $\bigcap_{i=1}^j V(a_i, n_i) \subseteq G$, for some $a_i \in \mathcal{B}, n_i, j \in N$.

The set $B_\varepsilon^a(0)$ is an open set containing 0, for any a in a Hamel basis $\mathcal{B}$ of Ω and $\varepsilon > 0$.

Therefore $\exists c > 0$ such that $\subseteq t B_\varepsilon^a(0), \forall t \geq c$. So,

$$\bigcap_{i=1}^j V(a_i, n_i) \subseteq t B_\varepsilon^a(0), \forall t \geq c.$$

Therefore, for all $t \geq c, \bigcap_{i=1}^j V(a_i, n_i) \subseteq t B_\varepsilon^a(0)$,

i.e. $\bigcap_{i=1}^j \frac{1}{t} V(a_i, n_i) \subseteq B_\varepsilon^a(0)$. Specifically, for $t = c$, thus

$$\frac{1}{c} V(a_i, n_i) = B_{\frac{1}{n_i c}}^{a_i}(0) \text{ and } \bigcap_{i=1}^j B_{\frac{1}{n_i c}}^{a_i}(0) \subseteq B_\varepsilon^a(0)$$

If $s \in \Omega$ and $\{s_n\}$ is a sequence in Ω such that $\lim_{n \rightarrow \infty} \|s - s_n, a_i\| = 0, \forall i = 1, 2, \dots, j$. Then there exists $k_i \in N$ such that $q_s(s - s_n, a_i) < \frac{1}{n_i c}, n \geq k_i$, where $i = 1, 2, \dots, j$

Take $k = \sup\{k_1, k_2, \dots, k_j\}$, Then, for all $n \geq k$, hence $s - s_n \in \bigcap_{i=1}^j B_{\frac{1}{n_i c}}^{a_i}(0) \subseteq B_\varepsilon^a(0)$.

Therefore, $q_s(s - s_n, a) < \varepsilon$ for all $n \geq k$ and $\lim_{n \rightarrow \infty} q_s(s - s_n, a_i) = 0$. Since $a \in \mathcal{B}$. This holds for all $a \in \Omega$. Thus, Ω satisfies property P_j .

From Theorems 3.21 and 3.22, a necessary and sufficient condition for normability is given below.

3.23. Theorem: A q_2 -normed space satisfies the property (P_j) for some $j \in N$ if and only if it is normable.

Hence, from Remark 3.20 and Theorem 3.23, we have the following:

3.24. Corollary: Every finite-dimensional q_2 -normed space is normable.

4. Norms by q_2 -norm

It can derive a norm from a normable q_2 -norm such that both have the same topology. Start with the following result:

4.1. Proposition: Let (Ω, q_s) be a q_2 -normed space and a linearly independent set $\{a_1, a_2, \dots, a_m\} \subseteq \Omega$. Let $\| \cdot \|_p : \Omega \rightarrow \mathbb{R}$, by $\|s\|_p = k(\sum_{i=1}^m [q_s(s, a_i)]^p)^{\frac{1}{p}}$, $\forall s \in \Omega$ where $1 \leq p < \infty$, and $\|s\|_\infty = \sup\{q_s(s, a_i), \forall s \in \Omega, i = 1, 2, \dots, m\}$, where $p = \infty$. Then all these norms are equivalent.

Proof: The outlines and some details of proof are similar to those mentioned in ^{10, 27}. So the details are deleted.

Another interesting result is

4.2. Theorem: Let (Ω, q_s) be a q_2 -normed space. The following are equivalent:

- Ω satisfies property (P_j) .
- The topology of (Ω, q_s) coincides with the topology of the norm defined by, $\|s\|_\infty = \sup\{q_s(s, a_1), q_s(s, a_2), \dots, q_s(s, a_j)\}$ for all $s \in \Omega$ and a linearly independent set $\{a_1, a_2, \dots, a_j\}$.

Proof: By assumption, there exist linearly independent elements $a_1, a_2, \dots, a_j$ of Ω such that $q_s(s - s_n, a_i) \rightarrow 0$ for all $i = 1, 2, \dots, j$

Implies $s_n \rightarrow s$ with respect to the q_2 -normed on Ω . Then

$$\|s\|_\infty = \sup\{q_s(s, a_1), q_s(s, a_2), \dots, q_s(s, a_j)\}, \forall s \in \Omega$$

The sequence $\{s_n\}$ in Ω such that $q_s(s - s_n, a) \rightarrow 0, \forall a \in \Omega$

Iff that $q_s(s - s_n, a_i) \rightarrow 0, \forall i = 1, 2, \dots, j$. If $\|s - s_n\|_\infty \rightarrow 0$

Therefore, a sequence $\{s_n\}$ converges in $(\Omega, \| \cdot \|_\infty)$ if and only if it converges in (Ω, q_s) .

On the other side, (Ω, q_s) satisfies the property (P_j) for some $j \in N$. So (Ω, q_s) is normable.

Therefore, both (Ω, q_s) and $(\Omega, \| \cdot \|_\infty)$ give the same topology.

Conversely, both (Ω, q_s) and $(\Omega, \| \cdot \|_\infty)$ give the same topology where,

$$\|s\|_\infty = \sup\{q_s(s, a_1), q_s(s, a_2), \dots, q_s(s, a_j)\} \text{ for all } s \in \Omega \text{ and some linearly independent elements } a_1, a_2, \dots, a_j. \text{ The sequence } \{s_n\} \text{ from } \Omega \text{ such that } q_s(s - s_n, a_i) \rightarrow 0 \text{ for all } i = 1, 2, \dots, j$$

Then it follows that $\|s - s_n\|_\infty \rightarrow 0$ and $q_s(s - s_n, a) \rightarrow 0$ for all $a \in \Omega$.

Hence (Ω, q_s) satisfies the property (P_j) .

4.3. Remark: The usefulness of Theorem 3.23 to construct a norm for a normable q_2 -normed space, and Theorem 4.2, is useful to ensure the normability of a q_2 -normed space. Therefore, two Theorems may be complements of each other.

4.4. Proposition: Let (Ω, q_s) be a finite-dimensional q_2 -normed space and a linearly independent vectors b_1, b_2 from Ω . Then, the topology of the norm $\| \cdot \|_\infty$ on Ω defined by $\|s\|_\infty = \sup\{q_s(s, b_1), q_s(s, b_2)\}, \forall s \in \Omega$ is the same as the topology of (Ω, q_s) .

Proof: Ω is finite-dimensional. So, (Ω, q_s) is normable. Now both $\|\cdot\|$ and $\|\cdot\|_\infty$ are a norm on Ω . But all norms over a finite-dimensional vector space are equivalent. Therefore, $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent. Hence, the topology of $\|\cdot\|_\infty$ is the same as the topology of the (Ω, q_s) .

4.5. Corollary: Every q_2 -norm over a finite-dimensional vector space gives the same topology.

4.6. Corollary: A finite-dimensional q_2 -normed space satisfies the property (P_2) .

4.7. Theorem: Let (Ω, q_s) be a q_2 -normed space and $\|\cdot\|_\infty$ be the derived norm. Then the convergence of sequences in (Ω, q_s) and $(\Omega, \|\cdot\|_\infty)$ is equivalent if and only if both the spaces have the same topology.

Proof: If (Ω, q_s) and $(\Omega, \|\cdot\|_\infty)$ have the same topology, then the convergence of sequences is equivalent.

Conversely, the convergence of sequences for (Ω, q_s) and $(\Omega, \|\cdot\|_\infty)$ are equivalent. A sequence $\{s_n\}$ in Ω and $s \in \Omega$. Then, $s_n \rightarrow s$ in $(\Omega, q_s) \Leftrightarrow s_n \rightarrow s$ in $(\Omega, \|\cdot\|_\infty)$

i.e. $\lim_{n \rightarrow \infty} q_s(s - s_n, a) = 0, \forall a \in \Omega \Leftrightarrow \lim_{n \rightarrow \infty} \|s - s_n\|_\infty = 0$.

However, the definition of $\|\cdot\|_\infty$ concludes the existence of linearly independent elements $a_1, a_2, \dots, a_j$ such that for any sequence $\{s_n\}$ in Ω and $s \in \Omega$. The following holds, $\lim_{n \rightarrow \infty} \|s - s_n\|_\infty = 0 \Leftrightarrow \lim_{n \rightarrow \infty} q_s(s - s_n, a_i) = 0$ for all $i = 1, 2, \dots, j$.

Also, $\lim_{n \rightarrow \infty} q_s(s - s_n, a) = 0$ for all $a \in \Omega$ if and only if $\lim_{n \rightarrow \infty} q_s(s - s_n, a_i) = 0$ for all $i = 1, 2, \dots, j$

So, (Ω, q_s) satisfies property (P_j) . Thus, by Theorem (4.2), the (Ω, q_s) and $(\Omega, \|\cdot\|_\infty)$ have the same topology.

Finally, Theorems (3.23-4.2-4.7) can be summarized in the following:

4.8. Theorem: Let (Ω, q_s) be a q_2 -normed space and $\|\cdot\|_\infty$ be the derived norm, then the following are equivalent,

- The convergence of sequences in (Ω, q_s) and $(\Omega, \|\cdot\|_\infty)$ are equivalent,
- (Ω, q_s) and $(\Omega, \|\cdot\|_\infty)$ have the same topology,
- (Ω, q_s) satisfies property (P_j) for some $j \in \mathbb{N}$,
- (Ω, q_s) is normable.

Clearly indicates that convergence of sequences with respect to the q_2 -normed and its derived norm are not equivalent if the q_2 -normed space is non-normable.

5. Conclusion

This paper recalled the concept of q_2 -normed spaces, which is a generalization of 2-normed spaces, and investigated some of their fundamental properties. It was shown that the topology induced by a q_2 -normed form of a (TVS) generated by a separating family of q -seminorms and sufficient conditions for metrizability were derived. Moreover, necessary and sufficient conditions for the normability of q_2 -normed spaces were obtained. In addition to current results, there will be a useful framework for future studies in operator theory, fixed point theory, and best proximity point results in q_2 -normed spaces. Since the balls in this space are not necessarily convex, the paper opens the door to an important question: How will the many facts that depend on convexity be handled?

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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