



New Distribution with Statistical Properties and Classical Estimation Method

Esraa Hussein Abbas^{1*} , Rehab Noori Shalan² , and Umar Yusuf Madaki³ 

^{1,2}Department of Mathematics, College of Sciences, University of Baghdad, Baghdad, Iraq.

³Department of Mathematics and Statistics, Yobe State University, Damaturu, Nigeria.

*Corresponding Author

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Abstract:

In the present study, we propose a new model, identified as the Shanker Rayleigh distribution, which is derived by combining the survival functions of the classical Rayleigh and Shanker distributions, both of which have a scale parameter only. We developed this hybrid model to offer the additional modeling flexibility required for complex lifespans that single-unit distributions cannot accurately represent. Theoretical characteristics of the distribution (probability density function, cumulative distribution function, survival, and hazard rate function) are derived in closed form. In addition, important distributional properties, namely the raw and central moments, moment-generating function, and quantile function, are methodically examined and developed from two statistical methods ,maximum likelihood estimation (MLE) and ordinary least squares (OLS), for estimating the parameter of the proposed model, and their efficiency was assessed by applying them to real data to demonstrate the new models' ability to better apply the data. In addition, this proposal was applied for scientific purposes through the analysis of real data.

Keywords: Shanker distribution, Rayleigh distribution, survival function, moment-generating function, quantile function.

1. Introduction:

In reality, the finite combination models are major in stochastic evaluation through the ever-rising ratio corresponding to articles on combination applications manifested in the stochastic and universal scholarly work.¹ In current years, a few research reports and scholarly work have examined various modern combination models. Employing mixed allocation, suitable modeling of diverse complex systems can be done². The application of combination distributions dates back to the late 19th century. Early research relied on these models to analyze empirical data; for instance, Newcomb (1886) utilized a mixture of normal distributions to model errors in astronomical observations³.

All of this is done using the data available to the researcher about the properties and nature of the work of each distribution⁴ The goal of combining probability distributions is to obtain a single refined distribution that integrates information about the same quantity from multiple original distribution⁵ Gaussian mixture forms are widely used in observations, mining, pattern identification, and machine learning, together with statistical analysis. In statistical analysis. In many implementations, their parameters are determined by statistical analysis. In many implementations by MLE, typically using the EM technique⁶ Researchers have studied mixing statistical distributions; for instance: in 2015⁷ proposed modelling lifetime data with a modern one-parameter lifetime allocation, the "Shanker distribution", by combining this "gamma(2, β) distribution" with the "exponential distribution".

In (2015), the exponentiated Lindley distribution was introduced for lifetime data estimation under progressive censoring⁸. It explains⁹ how to derive mixture distributions using moment-generating functions (MGF) and characteristic functions. Allocation for the scheme and the behavior regarding modest & medium deviations along with the (PD) scheme, the behavior

concerning ample deviations, propose ¹⁰new distributions with characteristics of (R-AP) in the previous study.

In (2022)¹¹ they undertook a new mixture of two Beta Exponential distributions and an income distribution. Previous research introduced ¹² the method to facilitate the integration of multidimensional data. It focuses on the probabilistic integration of conflicting and inaccurate data within a statistical framework to calculate "true values" and mixed distribution models¹³ to improve the fit of wind speed data using empirical search algorithms. In addition to distribution¹⁴ Bayesian Model Averaging integrates multiple predictive distributions and improves probabilistic accuracy in (2014)¹⁵.

It specifies how to create data that can be used over time, using finite admixture models to estimate their proportions within population groups. It has been suggested¹⁶new distributions are proposed within the Mixture Models framework, and the EM algorithm is used to estimate parameters and analyze skewed data, as well as¹⁷ considered a non-prejudiced assessment of some accuracy metrics based on a combination model of two exp. Explores¹⁸ the integration of Gaussian Mixture models in the context of decentralized Bayesian integration and the improvement of numerical methods for integrating distributions. The theoretical framework is designed for the mixture to accommodate heavy-tailed and jumbled distributions ^{19,20}.

Suggestion of the Copula Density Using the wavelet transform²¹; also, estimates the scale parameter for the theoretical distribution. In this paper, we introduce a new mixture of continuous distributions, which we call the 'Shanker-Rayleigh distribution,' by combining the 'Shanker probabilistic model' and the 'Rayleigh+ probabilistic model' according to their survival roles for both distributions. Section 2 describes how we mixed the 'Shanker probabilistic model' with the 'Rayleigh probabilistic model' to obtain such a 'Shanker-Rayleigh distribution' (SHRD) and introduces the basic statistical functions such as the probability density function, the cumulative density function, the survival function, and the hazard rate function. In addition to the shape of the distribution.

Section 3 presents the mathematical properties of the Shanker-Rayleigh distribution, including the mode, quantile function, moments about the origin, characteristic function, moment generation function, factorial moment generation function, and mean time to failure. Section 4 two methods for non-Bayesian (classical) estimation, namely the Maximum Likelihood (MLE) method and the Ordinary Least Squares (OLS) method, which were provided using the Newton-Raphson (NR) method, and the numerical values of the parameters were calculated using real data. and presents numerical results and comparisons of statistical methods using mean square error and simulation-based methods using the Monte Carlo technique, in addition to the real data and its application.

2. Materials and Method

2.1. Structure of the New Shanker-Rayleigh distribution

The pdf, cdf and survival function of the Shanker distribution ²² are

$$f_{sh}(q, \beta) = \frac{\beta^2}{\beta^2 + 1} (\beta + q) e^{-\beta q} \quad (1)$$

where $x > 0$ and $q > 0, \beta > 0$ is scale parameter

$$F_{sh}(q, \beta) = 1 - \frac{(\beta^2 + 1) + \beta q}{\beta^2 + 1} e^{-\beta q} \quad (2)$$

$$S_{sh}(q) = \frac{(\beta^2 + 1) + \beta q}{\beta^2 + 1} e^{-\beta q} \quad (3)$$

The PDF, CDF, and survival function of the Rayleigh distribution²³ are

$$f_R(v, \gamma) = \frac{v}{\gamma^2} e^{-\frac{v^2}{2(\gamma^2)}} \quad (4)$$

$$F_R(v, \gamma) = 1 - e^{-\frac{v^2}{2(\gamma^2)}} \quad (5)$$

$$S_R(v, \gamma) = e^{-\frac{v^2}{2(\gamma^2)}} \quad (6)$$

Let q and v are independent random variables $X = \min(Q, V)$ than

$$s(x) = p_{r(X=x)} = p_r(\min(q, v) > X),$$

From **Equations 3 and 6**, we obtain the survival function:

$$S_{(SHRD)}(x; \beta, \gamma) = \frac{(\beta^2+1)+\beta x}{\beta^2+1} e^{-\left(\beta x + \frac{x^2}{2(\gamma^2)}\right)} \quad (7)$$

Where $x > 0$, and $\beta > 0$ is the scale parameter,

The CDF and PDF of the new (SHRD) distribution is :

$$F_{(SHRD)}(x; \beta, \gamma) = 1 - \frac{(\beta^2+1)+\beta x}{\beta^2+1} e^{-\left(\beta x + \frac{x^2}{2(\gamma^2)}\right)} \quad (8)$$

$$f_{(SHRD)}(x; \beta, \gamma) = \left(\frac{(\beta^2+1)+\beta x}{\beta^2+1} \cdot \left(\beta + \frac{2x}{2\gamma^2} \right) - \frac{\square}{1+\beta^2} \right) \quad (9)$$

By using l'Hôpital's Rule for the above limit,

$$\int_0^\infty f(x) dx = -\lim_{y \rightarrow \infty} \left(\frac{\beta}{(\beta^2+1)\beta x + \frac{y}{2(\gamma^2)}} - 1 \right) = 1$$

$$h(x) = \frac{f(x)}{s(x)} = \left(\beta + \frac{x}{\gamma^2} \right) - \frac{\beta}{(\beta^2+\beta x+1)} \quad (10)$$

2.2. The shape of the new Shanker Rayleigh distribution

Identifying the pattern of the SHRD facilitates a mathematical model of distribution functions in handling data to understand this mathematically, notably through the limit values. of the probability density and hazard functions($x \rightarrow 0, x \rightarrow \infty$)

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \left((\beta^2 + \beta x + 1) \left(\beta + \frac{x}{\gamma} \right) - \beta \right) \frac{1}{\beta^2 + 1} e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} \\ &= \frac{\beta^3}{\beta^2 + 1} \\ \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \left(\frac{(1 + \beta^2 + \beta x) \left(\frac{x}{\gamma^2} + \beta \right)}{\left(\frac{1}{\beta^2 + 1} \right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)}} - \frac{\beta}{\left(\frac{1}{\beta^2 + 1} \right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)}} \right) \\ &\text{If } \gamma > 0, \text{ applied L'Hôpital Rule} \\ \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{\gamma^2} \right) (\beta^2 + \beta x + 1) + \left(\beta + \frac{x}{\gamma^2} \right) \beta}{(\beta^2 + 1) \left(\beta + \left(\frac{x}{\gamma} \right) \right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)}} = 0 \\ \lim_{x \rightarrow 0} h(x) &= \lim_{x \rightarrow 0} \left(\left(\beta + \frac{x}{\gamma} \right) - \frac{\beta}{\beta^2 + \beta x + 1} \right) = \frac{\beta^3}{\beta^2 + 1} \\ &= \lim_{x \rightarrow \infty} \left(\left(\beta + \frac{x}{\gamma} \right) - \frac{\beta}{\beta^2 + \beta x + 1} \right) = \infty \end{aligned}$$

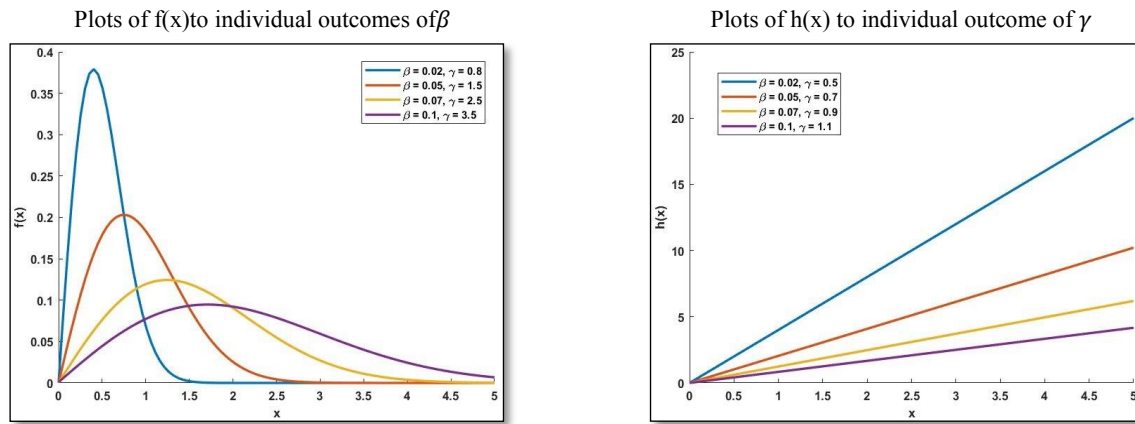


Figure 1. The shape of PDF & h(x) of new Shanker Rayleigh distribution with different values of β & γ .

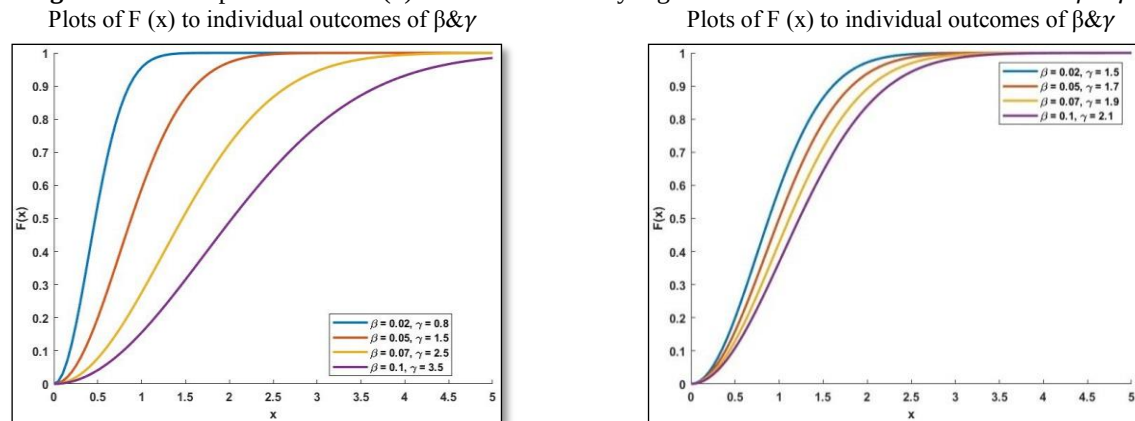


Figure 2. The shape of the CDF of the new Shanker Rayleigh distribution with different values of β & γ .

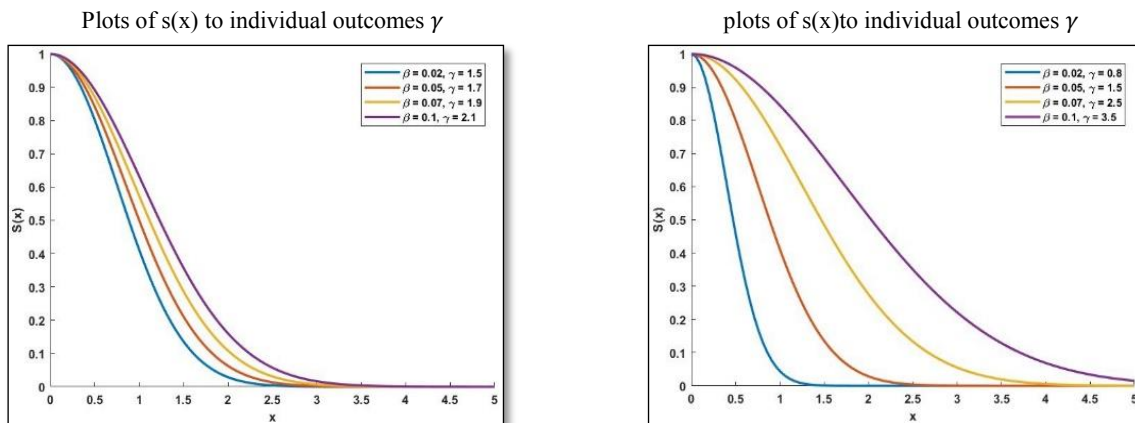


Figure 3. The shape of S(x) of the new Shanker Rayleigh distribution with different values β & γ

3. Mathematical Properties of the Shanker Rayleigh Distribution

3.1. Mode:

$$f(x) = \left(\frac{\beta^2 + \beta x + 1}{\beta^2 + 1} \left(\beta + \frac{x}{\gamma^2} \right) - \frac{\beta}{\beta^2 + 1} \right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2} \right)}$$

$$\frac{\partial f(x)}{\partial x} = \left(\frac{\beta^2 + \beta x + 1}{\beta^2 + 1} \left(\beta + \frac{x}{\gamma^2} \right) - \frac{\beta}{\beta^2 + 1} \right) \left(- \left(\beta + \frac{x}{\gamma^2} \right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2} \right)} \right) + e^{-\left(\beta x + \frac{x^2}{2\gamma^2} \right)} \left(\frac{\beta^2 + \beta x + 1}{\beta^2 + 1} \cdot \frac{1}{\gamma^2} \right) + \frac{\beta}{\beta^2 + 1} \cdot \left(\beta + \frac{x}{\gamma^2} \right)$$

$$\frac{1}{\beta^2 + 1} \neq 0, e^{-\left(\beta x + \frac{x^2}{2\gamma^2} \right)} \neq 0 \text{ for each selected value of } \beta, \gamma \text{ and } x, \text{ this leads to}$$

$$1 - \left(\beta + \frac{x}{\gamma^2} \right) = 0$$

Thus, we have two cases:

Case 1 (Mode):

$$1 - \left(\beta + \frac{x}{\gamma^2} \right) = 0$$

$$1 - \beta - \frac{x}{\gamma^2} = 0$$

$$x = \gamma^2(1 - \beta)$$

Likewise, we have the following cases:

$$2((\beta^2 + 1) + \beta x) - \left(\beta + \frac{x}{\gamma} \right)^2 \cdot \frac{1}{\gamma^2} + (\beta^2 + 1 + \beta) - \beta = 0$$

$$2((\beta^2 + 1) + \beta x) - \frac{\left(\beta + \frac{x}{\gamma} \right)^2}{\gamma^2} + (\beta^2 + 1) = 0$$

$$2\beta^2 + 2 + 2\beta x + \beta^2 + 1 = 3\beta^2 + 3 + 2\beta x$$

$$\left(\beta + \frac{x}{\gamma} \right)^2 = \beta^2 + \frac{2\beta x}{\gamma} + \frac{x^2}{\gamma^2} \quad \div \gamma^2$$

$$\frac{\beta^2}{\gamma^2} + \frac{2\beta x}{\gamma^3} + \frac{x^2}{\gamma^4}$$

$$(3\beta^2 + 3 + 2\beta x) - \left(\frac{\beta^2}{\gamma^2} + \frac{2\beta x}{\gamma^3} + \frac{x^2}{\gamma^4} \right) = 0$$

Since

$$x^2 \Rightarrow -\frac{x^2}{\gamma^4}$$

$$x \Rightarrow 2\beta x - \frac{2\beta x}{\gamma^3}$$

$$\text{constants} \Rightarrow (2\beta^2 + 2) - \frac{\beta^2}{\gamma^2}$$

$$ax^2 + bx + c$$

Given that:

$$a = -\frac{1}{\gamma^4}$$

$$b = 2\beta \left(1 - \frac{1}{\gamma^3} \right)$$

$$c = 3\beta^2 + 3 - \frac{\beta^2}{\gamma^2}$$

$$x = \frac{-2\beta \left(1 - \frac{1}{\gamma^3} \right) \pm \sqrt{2\beta \left(1 - \frac{1}{\gamma^3} \right)^2 - 4 \left(-\frac{1}{\gamma^4} \right) \left(3\beta^2 + 3 - \frac{\beta^2}{\gamma^2} \right)}}{2 \left(-\frac{1}{\gamma^4} \right)}$$

$$x = \frac{-2\beta \left(1 - \frac{1}{\gamma^3} \right) \pm \sqrt{2\beta \left(1 - \frac{1}{\gamma^3} \right)^2 - 4 \left(-\frac{1}{\gamma^4} \right) \left(3\beta^2 + 3 - \frac{\beta^2}{\gamma^2} \right)}}{2 \left(-\frac{1}{\gamma^4} \right)}$$

$$x = -\frac{\gamma^4}{2} \left(-2\beta \left(1 - \frac{1}{\gamma^3} \right) \pm \sqrt{2\beta \left(1 - \frac{1}{\gamma^3} \right)^2 - 4 \left(-\frac{1}{\gamma^4} \right) \left(3\beta^2 + 3 - \frac{\beta^2}{\gamma^2} \right)} \right)$$

3.2. Quantile Function

Let $F(x)=v, v \sim v(0,1)$, then $F(x)=v, x = F^{-1}(v)$

$$v = 1 - \frac{\beta^2 + \beta x + 1}{\beta^2 + 1} \cdot e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)}$$

$$v - 1 = -\frac{\beta^2 + \beta x + 1}{\beta^2 + 1} \cdot e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)}$$

$$1 - v = \frac{\beta^2 + \beta x + 1}{\beta^2 + 1} \cdot e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)}$$

$$\ln(1 - v) = \ln(\beta^2 + \beta x + 1) - \ln(\beta^2 + 1) - \left(\beta x + \frac{x^2}{2\gamma^2}\right)$$

$$\ln(1 - v) + \beta x + \frac{x^2}{2\gamma^2} + \ln(\beta^2 + 1) = \ln(\beta^2 + \beta x + 1)$$

$$\beta x + \frac{x^2}{2\gamma^2} + \ln(1 - v) + \ln(\beta^2 + 1) = \ln(\beta^2 + 1) + \frac{\beta x}{\beta^2 + 1}$$

by the Taylor Approximation:

$$\ln(\beta^2 + \beta x + 1) = \ln(\beta^2 + 1) + \frac{\beta x}{\beta^2 + 1}$$

$$\beta x + \frac{x^2}{2\gamma^2} + \ln(1 - v) = \frac{\beta x}{\beta^2 + 1}$$

$$\beta x + \frac{x^2}{2\gamma^2} - \frac{\beta x}{\beta^2 + 1} + \ln(1 - v) = 0$$

$$\frac{x^2}{2\gamma^2} + \beta x \left(1 - \frac{1}{\beta^2 + 1}\right) + \ln(1 - v) = 0$$

$$\frac{x^2}{2\gamma^2} + \frac{\beta^3}{\beta^2 + 1} x + \ln(1 - v) = 0$$

$$x = -\frac{\beta^3}{\beta^2 + 1} \pm \sqrt{\frac{\beta^3}{\beta^2 + 1}^2 - \frac{2}{\gamma^2} \ln(1 - v)}$$

3.3. Characteristic function

$$f(x) = \left(1 + \frac{\beta x}{1 + \beta^2} \left(\frac{x}{\gamma^2} + \beta\right) - \frac{\beta}{1 + \beta^2}\right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)}, \quad x > 0$$

$$\varphi_p(ip) = E(e^{itp}) = \int_0^\infty e^{itp} f(p) dp$$

$$\varphi_p(ip) = \int_0^\infty e^{itp} \left(\left(1 + \frac{\beta p}{1 + \beta^2}\right) \left(\frac{p}{\gamma^2} + \beta\right) - \frac{\beta}{1 + \beta^2} \right) e^{-\left(\beta p + \frac{p^2}{2\gamma^2}\right)} dp$$

$$\square \text{ et } e^{-\left(\frac{p^2}{2\gamma^2}\right)} = \sum_0^\infty \frac{(-1)^n}{n!} \frac{p^{2n}}{(2\gamma^2)^n} \Rightarrow \sum_0^\infty \frac{(-1)^n}{n!(2\gamma^2)^n} p^{2n}$$

$$\varphi_p(ip) = \sum_0^\infty \frac{(-1)^n}{n!(2\gamma^2)^n} \left(\int_0^\infty e^{itp} \left(\beta + \frac{p}{\gamma^2} + \frac{\beta^2 p}{\beta^2 + 1} + \frac{\beta}{(\beta^2 + 1)\gamma^2} p^2 - \frac{\beta}{\beta^2 + 1} \right) e^{-\beta p} \cdot p^{2n} dp \right)$$

$$= \sum_0^\infty \frac{(-1)^n}{n!(2\gamma^2)^n} \left(\int_0^\infty \beta p^{2n} e^{-(\beta - it)p} dp + \int_0^\infty \frac{1}{\gamma^2} p^{2n+1} e^{-(\beta - it)p} dp + \right.$$

$$\left. \int_0^\infty \left(\frac{\beta^2}{\beta^2 + 1}\right) p^{2n+1} e^{-(\beta - it)p} dp + \right.$$

$$\left. \int_0^\infty \frac{\beta}{(\beta^2 + 1)\gamma^2} p^{2n+2} e^{-(\beta - it)p} dp - \int_0^\infty \frac{\beta}{\beta^2 + 1} p^{2n} e^{-(\beta - it)p} dp \right)$$

$$\text{Let } (\beta - it)p = r, \text{ and } p = \frac{r}{\beta - it} \text{ then } dp = \frac{dr}{\beta - it}$$

$$\begin{aligned} \varphi_p(it) &= \sum_0^\infty \frac{(-1)^n}{n!(2\gamma^2)^n} \left(\left(\frac{\beta}{(\beta-it)^{2n+1}} \right) \int_0^\infty r^{2n} e^{-r} dr + \left(\frac{1}{\gamma^2(\beta-it)^{2n+2}} \right) \int_0^\infty r^{2n+1} e^{-r} dr + \right. \\ &\quad \left(\frac{\beta^2}{(\beta^2+1)(\beta-it)^{2n+2}} \right) \int_0^\infty r^{2n+1} e^{-r} dr + \left(\frac{\beta}{(\beta^2+1)\gamma^2(\beta-it)^{2n+3}} \right) \int_0^\infty r^{2n+2} e^{-r} dr - \\ &\quad \left. \left(\frac{\beta}{(\beta^2+1)(\beta-it)^{2n+1}} \right) \int_0^\infty r^n e^{-r} dr \right) \\ \varphi_p(it) &= \\ \sum_0^\infty \frac{(-1)^n}{n!(2\gamma^2)^n} &\left(\frac{\beta \cdot \Gamma(1+2n)}{(\beta-it)^{2n+1}} \right) + \left(\frac{1}{\gamma^2} \cdot \frac{\Gamma(2+2n)}{(\beta-it)^{2n+2}} \right) + \frac{\beta \cdot \Gamma(2n+3)}{(\beta^2+1)\gamma^2(\beta-it)^{2n+3}} + \frac{\beta^2 \cdot \Gamma(2+2n)}{(\beta^2+1)(\beta-it)^{2n+2}} - \\ &\frac{\beta \cdot \Gamma(2n+1)}{(\beta^2+1)(\beta-it)^{2n+1}} \end{aligned}$$

3.4. Moment of the Origin

$$\begin{aligned} M'r(x) &= \int_0^\infty x^r f(x) dx \\ &= \\ \int_0^\infty x^r &\left(\beta e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} + \frac{x}{\gamma^2} e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} + \frac{\beta^2 x}{\beta^2+1} e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} + \frac{\beta x^2}{1+\beta^2+\gamma^2} e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} - \right. \\ &\left. \frac{\beta}{1+\beta^2} e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} \right) \end{aligned}$$

$$\text{Let } e^{-\frac{x^2}{2\gamma^2}} = \sum_{j=0}^\infty \frac{(-1)^j}{j!(2\gamma^2)^j} \cdot x^{2j} \quad (\text{prudnikov.1986})$$

$$\begin{aligned} \int_0^\infty x^r f(x) dx &= \\ \sum_{j=0}^\infty \frac{(-1)^j}{j!(2\gamma^2)^j} &\left(\int_0^\infty \beta x^{r+2j} e^{-\beta x} dx + \int_0^\infty \frac{1}{\gamma^2} x^{2j+1+r} e^{-\beta x} dx + \int_0^\infty \frac{\beta^2}{\beta^2+1} x^{r+2j+1} e^{-\beta x} dx + \right. \\ &\left. \int_0^\infty \frac{\beta}{\beta^2+1+\gamma^2} x^{2+r+2j} e^{-\beta x} dx - \int_0^\infty \frac{\beta}{\beta^2+1} x^{2j+r} e^{-\beta x} dx \right) \end{aligned}$$

$$\beta x = m, \text{ then } x = \frac{m}{\beta}, dx = \frac{dm}{\beta} \text{ since}$$

$$\begin{aligned} M''r(x) &= \sum_0^\infty \frac{(-1)^j}{j!(2\gamma^2)^j} \left(\left(\frac{1}{(\beta)^{r+2j+1}} \right) \int_0^\infty y^{r+2j} e^{-y} dy + \left(\frac{1}{\gamma^2(\beta)^{r+2j+2}} \right) \int_0^\infty y^{2j+r} e^{-y} dy + \right. \\ &\left(\frac{1}{(\beta^2+1)(\beta)^{r+2j}} \right) \int_0^\infty y^{2j+r+1} e^{-y} dy + \left(\frac{1}{(\beta^2+1)\gamma^2(\beta-it)^{r+2j+2}} \right) \int_0^\infty y^{2j+r+2} e^{-y} dy - \\ &\left. \left(\frac{1}{(\beta^2+1)(\beta)^{2j+r}} \right) \int_0^\infty y^{2j+r} e^{-y} dy \right) \\ &= \left(\frac{\Gamma(1+r+2j)}{\beta^{r+2j+1}} + \frac{\frac{1}{\gamma^2} \cdot \Gamma(2+r+2j)}{\beta^{r+2j+2}} + \frac{\Gamma(2+r+2j) - \Gamma(1+r+2j)}{(\beta^2+1) \cdot \beta^{r+2j}} + \frac{\Gamma(3+r+2j)}{(\beta^2+1)\gamma^2(\beta)^{r+2j+2}} \right) \end{aligned}$$

As a direct result, it can be found $E(x) = M_1'(x)$, $E(x^2) = M_2'(x)$, and $\text{var}(x)$ as follows:

$$E(x) = \sum_0^\infty \frac{(-1)^j}{j!(2\gamma^2)^j} \left(\frac{\Gamma(2j+2)}{\beta^{2j+2}} + \frac{\frac{1}{\gamma^2} \cdot \Gamma(2j+3)}{\beta^{2j+3}} + \frac{\Gamma(2j+3) - \Gamma(2j+2)}{(\beta^2+1) \cdot \beta^{2j+2}} + \frac{\Gamma(2j+4)}{(\beta^2+1)\gamma^2(\beta)^{2j+3}} \right)$$

$$E(x^2) = \sum_0^\infty \frac{(-1)^j}{j!(2\gamma^2)^j} \left(\frac{\Gamma(2j+3)}{\beta^{2j+3}} + \frac{\frac{1}{\gamma^2} \cdot \Gamma(2j+4)}{\beta^{2j+4}} + \frac{\Gamma(2j+4) - \Gamma(2j+3)}{(\beta^2+1) \cdot \beta^{2j+3}} + \frac{\Gamma(2j+5)}{(\beta^2+1)\gamma^2(\beta)^{2j+4}} \right)$$

$$\text{var}(x) = M_2'(x) - (M_1'(x))^2$$

3.5. Moment-Generating Function

$$\begin{aligned} M_x(t) &= E(e^{tx}) = \int_0^\infty e^{tx} f(x) dx \\ &= \int_0^\infty e^{tx} \left(1 + \frac{\beta x}{\beta^2+1} \left(\beta + \frac{x}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} dx \end{aligned}$$

$$\text{Let } e^{\frac{x^2}{2\gamma^2}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2\gamma^2)^k} \cdot x^{2k}$$

$$M_X(t) =$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2\gamma^2)^k} \left(\int_0^{\infty} \beta x^{2k} e^{-(\beta-t)x} dx + \int_0^{\infty} \frac{1}{\gamma^2} x^{2k+1} e^{-(\beta-t)x} dx + \right. \\ \left. \int_0^{\infty} \left(\frac{\beta^2}{\beta^2+1} \right) x^{2k+1} e^{-(\beta-t)x} dx + \int_0^{\infty} \frac{\beta}{(\beta^2+1)\gamma^2} x^{2k+2} e^{-(\beta-t)x} dx - \int_0^{\infty} \frac{\beta}{\beta^2+1} x^{2k} e^{-(\beta-t)x} dx \right)$$

$$\text{Let } (\beta - t)x = y \text{ as well as } x = \frac{y}{\beta - t} \text{ than } dx = \frac{dy}{\beta - t}$$

$$M_X(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2\gamma^2)^k} \left(\left(\frac{\beta}{(\beta-t)^{2k+1}} \right) \int_0^{\infty} y^{2k} e^{-y} dy + \left(\frac{1}{\gamma^2 (\beta-t)^{2k+2}} \right) \int_0^{\infty} y^{2k+1} e^{-y} dy + \right. \\ \left(\frac{\beta^2}{(\beta^2+1)(\beta-t)^{2k+2}} \right) \int_0^{\infty} y^{2k+1} e^{-y} dy + \left(\frac{\beta}{(\beta^2+1)\gamma^2 (\beta-t)^{2k+3}} \right) \int_0^{\infty} y^{2k+2} e^{-y} dy - \\ \left(\frac{\beta}{(\beta^2+1)(\beta-t)^{2k+2}} \right) \int_0^{\infty} y^{2k+1} e^{-y} dy \right)$$

$$M_X(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2\gamma^2)^k} \left(\frac{\beta \cdot \Gamma(1+2k)}{(\beta-t)^{2k+1}} + \frac{1/\gamma^2 \cdot \Gamma(2k+2)}{(\beta-t)^{2k+2}} + \frac{\beta^2 \cdot \Gamma(2+2k)}{(1+\beta^2)(\beta-t)^{2k+2}} - \frac{\beta \cdot \Gamma(2k+2)}{(1+\beta^2)(\beta-t)^{2k+2}} + \right. \\ \left. \frac{\beta \cdot \Gamma(2k+3)}{(\beta^2+1)\gamma^2 (\beta-t)^{2k+3}} \right)$$

3.6. Factorial Moment-Generating Function

$$M_X(t) = E(t^x) = \int_0^{\infty} t^x f(x) dx$$

$$= \int_0^{\infty} e^{x \ln t} \left(1 + \frac{\beta x}{\beta^2+1} \left(\beta + \frac{x}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2} \right)} dx$$

$$= \int_0^{\infty} \left(1 + \frac{\beta x}{\beta^2+1} \left(\beta + \frac{x}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right) e^{-\left((\beta - \ln t)x + \frac{x^2}{2\gamma^2} \right)} dx$$

$$\text{Let } e^{\frac{x^2}{2\gamma^2}} = \sum_{h=0}^{\infty} \frac{(-1)^h}{h!(2\gamma^2)^h} \cdot x^{2h}$$

$$M_X(t) =$$

$$\sum_{h=0}^{\infty} \frac{(-1)^h}{h!(2\gamma^2)^h} \left(\int_0^{\infty} \beta x^{2h} e^{-(\beta - \ln t)x} dx + \int_0^{\infty} \frac{1}{\gamma^2} x^{2h+1} e^{-(\beta - \ln t)x} dx + \right. \\ \left. \int_0^{\infty} \frac{\beta^2}{\beta^2+1} x^{2h+1} e^{-(\beta - \ln t)x} dx - \int_0^{\infty} \frac{\beta}{\beta^2+1} x^{2h} e^{-(\beta - \ln t)x} dx + \int_0^{\infty} \frac{\beta}{(\beta^2+1)\gamma^2} x^{2h+2} e^{-(\beta - \ln t)x} dx \right)$$

$$\text{Let } (\beta - \ln t)x = y \text{ and } x = \frac{y}{\beta - \ln t} \text{ than } dx = \frac{dy}{\beta - \ln t}$$

$$M_X(t) = \sum_{h=0}^{\infty} \frac{(-1)^h}{h!(2\gamma^2)^h} \left(\frac{\beta \cdot \Gamma(2h+1)}{(\beta - \ln t)^{2h+1}} + \frac{1/\gamma^2 \cdot \Gamma(2h+2)}{(\beta - \ln t)^{2h+2}} + \frac{\beta^2 \cdot \Gamma(2h+2)}{(\beta^2+1)(\beta - \ln t)^{2h+2}} - \frac{\beta \cdot \Gamma(2h+1)}{(\beta^2+1)(\beta - \ln t)^{2h+2}} + \right. \\ \left. \frac{\beta \cdot \Gamma(2h+3)}{(\beta^2+1)\gamma^2 (\beta - \ln t)^{2h+3}} \right)$$

3.7. Mean Time to Failure

$$S_{(SHRD)}x = pr(T > x)$$

since $S_{(SHRD)}(x) \rightarrow 0$ at $t \rightarrow \infty$, than is given by:-

$$MTTE = \int_0^{\infty} S_{(SHRD)}x dx = \int_0^{\infty} \left(1 + \frac{\beta x}{\beta^2+1} \right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2} \right)} dx$$

$$\text{Recalling that, } e^{\frac{x^2}{2\gamma^2}} = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!(2\gamma^2)^v} \cdot x^{2v}$$

$$MTTE = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!(2\gamma^2)^v} \left(\int_0^{\infty} x^{2v} \cdot e^{-\beta x} dx + \frac{\beta}{\beta^2+1} \int_0^{\infty} x^{2v+1} \cdot e^{-\beta x} dx \right)$$

$$\text{Let } dx = y \text{ and } x = \frac{y}{\beta} \text{ than } dx = \frac{dy}{\beta}$$

$$\begin{aligned}
MTTE &= \sum_{v=0}^{\infty} \frac{(-1)^v}{v!(2\gamma^2)^v} \left(\int_0^{\infty} \left(\frac{y}{\beta}\right)^{2v} \cdot e^{-y} \frac{dy}{\beta} + \frac{\beta}{\beta^2+1} \int_0^{\infty} \left(\frac{y}{\beta}\right)^{2v+1} \cdot e^{-\beta x} \frac{dy}{\beta} \right) \\
MTTE &= \sum_{v=0}^{\infty} \frac{(-1)^v}{v!(2\gamma^2)^v} \left(\frac{1}{\beta^{2v+1}} \int_0^{\infty} y^{2\Box} \cdot e^{-y} dy + \frac{1}{(\beta^2+1)\beta^{2v+1}} \int_0^{\infty} y^{2v+1} \cdot e^{-\beta x} dy \right) \\
MTTE &= \sum_{v=0}^{\infty} \frac{(-1)^v}{v!(2\gamma^2)^v} \left(\frac{1}{\beta^{2v+1}} \cdot \Gamma(2v+1) + \frac{1}{(\beta^2+1)\beta^{2v+1}} \cdot \Gamma(2v+2) \right)
\end{aligned}$$

4. Estimation Methods and Data Study

4.1 Estimation Methods

4.1.1 Maximum Likelihood Estimation of SHRD

Parameter estimation is a basic step in statistical analysis. Of the various methods available, maximum likelihood estimation (MLE) is one of the most commonly used due to its mathematical rigor and broad applicability across different types of data and models.²⁴

$$f(x, \beta, \gamma) = \left(1 + \frac{\beta x}{\beta^2+1} \left(\beta + \frac{x}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right) e^{-\left(\beta x + \frac{x^2}{2\gamma^2} \right)}$$

$$L(\beta, \gamma, x_1, \dots, x_n) = \prod_{i=1}^n f(\beta, \gamma, x_i)$$

$$L(\beta, \gamma, x_i) = \prod_{i=1}^n \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right) e^{-\left(\beta x_i + \frac{x_i^2}{2\gamma^2} \right)}$$

$$\ln L(\beta, \gamma, x_i) = \ln \prod_{i=1}^n \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right) + \ln e^{-\sum_{i=1}^n \left(\beta x_i + \frac{x_i^2}{2\gamma^2} \right)}$$

$$\ln L(\beta, \gamma, x_i) = \sum_{i=1}^n \ln \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right) - \sum_{i=1}^n \left(\beta x_i + \frac{x_i^2}{2\gamma^2} \right)$$

The Newton-Raphson method is used to solve a nonlinear equation, and its steps are as follows:

$$\begin{bmatrix} \widehat{\beta}_{l+1} \\ \widehat{\gamma}_{l+1} \end{bmatrix} = \begin{bmatrix} \widehat{\beta}_l \\ \widehat{\gamma}_l \end{bmatrix} - j^{-1} \begin{bmatrix} \frac{\partial \ln L}{\partial \beta} \\ \frac{\partial \ln L}{\partial \gamma} \end{bmatrix} \text{ where } j = \begin{bmatrix} \frac{\partial^2 \ln L}{\partial^2 \beta} & \frac{\partial^2 \ln L}{\partial \beta \partial \gamma} \\ \frac{\partial^2 \ln L}{\partial \gamma \partial \beta} & \frac{\partial^2 \ln L}{\partial^2 \gamma} \end{bmatrix}$$

$$W(\beta) = \frac{\partial \ln L}{\partial \beta} = \sum_{i=1}^n \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right)^{-1} \left(\frac{\beta x_i}{\beta^2+1} + \left(\beta + \frac{x_i}{\gamma^2} \right) \left(\frac{x_i - \beta^2 x_i}{(\beta^2+1)^2} \right) - \left(\frac{1 - \beta^2}{(\beta^2+1)^2} \right) \right) - \sum_{i=1}^n x_i = 0 \quad (11)$$

$$M(\gamma) = \frac{\partial \ln L}{\partial \gamma} = \sum_{i=1}^n \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right)^{-1} \left(\frac{\beta x_i}{\beta^2+1} \cdot \frac{-2x_i}{\gamma^3} \right) + \sum_{i=1}^n \frac{x_i^2}{\gamma^3} \quad (12)$$

$$\begin{aligned}
\frac{\partial W(\beta)}{\partial \beta} &= \sum_{i=1}^n \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right)^{-1} \left(\left(\frac{x_i - \beta^2 x_i}{(\beta^2+1)^2} \right) - \frac{6\beta^2 x_i + 6\beta x_i}{(\beta^2+1)^2} \left(\beta + \frac{x_i}{\gamma^2} \right) + \left(\frac{x_i - \beta^2 x_i}{(\beta^2+1)^2} \right) - \right. \\
&\quad \left. \left(\frac{2\beta^3 - 6\beta}{(\beta^2+1)^3} \right) + \left(\frac{\beta x_i}{\beta^2+1} + \left(\beta + \frac{x_i}{\gamma^2} \right) \left(\frac{x_i - \beta^2 x_i}{(\beta^2+1)^2} \right) - \left(\frac{1 - \beta^2}{(\beta^2+1)^2} \right) \right) (-1) \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \right. \right. \\
&\quad \left. \left. \frac{\beta}{\beta^2+1} \right)^{-2} \left(\frac{\beta x_i}{\beta^2+1} + \left(\beta + \frac{x_i}{\gamma^2} \right) \left(\frac{x_i - \beta^2 x_i}{(\beta^2+1)^2} \right) - \left(\frac{1 - \beta^2}{(\beta^2+1)^2} \right) \right) \right) \quad (13)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial W(\beta)}{\partial \gamma} &= \sum_{i=1}^n \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right)^{-1} \left(\left(\frac{x_i - \beta^2 x_i}{(\beta^2+1)^2} \right) \cdot \frac{-2x_i}{\gamma^3} \right) + \left(\frac{\beta x_i}{\beta^2+1} + \left(\beta + \frac{x_i}{\gamma^2} \right) \left(\frac{x_i - \beta^2 x_i}{(\beta^2+1)^2} \right) - \right. \\
&\quad \left. \left(\frac{1 - \beta^2}{(\beta^2+1)^2} \right) \right) (-1) \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right)^{-2} \left(\left(\frac{x_i - \beta^2 x_i}{(\beta^2+1)^2} \right) \cdot \frac{-2x_i}{\gamma^3} \right) \quad (14)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial M(\gamma)}{\partial \gamma} &= \sum_{i=1}^n \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right)^{-1} \left(\frac{\beta x_i}{\beta^2+1} \cdot \frac{6x_i}{\gamma^4} \right) + \left(\frac{\beta x_i}{\beta^2+1} \cdot \frac{-2x_i}{\gamma^3} \right) (-1) \left(1 + \frac{\beta x_i}{\beta^2+1} \left(\beta + \right. \right. \\
&\quad \left. \left. \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2+1} \right)^{-2} \left(\frac{\beta x_i}{\beta^2+1} \cdot \frac{2x_i}{\gamma^3} \right) + \sum_{i=1}^n \frac{3x_i^2}{\gamma^4} \quad (15)
\end{aligned}$$

$$\frac{\partial M(\gamma)}{\partial \beta} = \sum_{i=1}^n \left(1 + \frac{\beta x_i}{\beta^2 + 1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2 + 1} \right)^{-1} \left(\frac{x_i - \beta^2 x_i}{(\beta^2 + 1)^2} \cdot \frac{-2x_i}{\gamma^3} \right) + \left(\frac{\beta x_i}{\beta^2 + 1} \cdot \frac{-2x_i}{\gamma^3} \right) \cdot (-1) \left(1 + \frac{\beta x_i}{\beta^2 + 1} \left(\beta + \frac{x_i}{\gamma^2} \right) - \frac{\beta}{\beta^2 + 1} \right)^{-2} \left(\frac{\beta x_i}{\beta^2 + 1} + \left(\beta + \frac{x_i}{\gamma^2} \right) \left(\frac{x_i - \beta^2 x_i}{(\beta^2 + 1)^2} \right) - \left(\frac{1 - \beta^2}{(\beta^2 + 1)^2} \right) \right) \quad (16)$$

Observe that: $\frac{\partial W(\beta)}{\partial \gamma} = \frac{\partial M(\gamma)}{\partial \beta}$ since this Jacobean matrix is symmetric

4.1.2 Ordinary Least Squares Estimation Method

The ordinary least squares method is one of the most frequently used procedures to estimate the parameters when the model is linear or nonlinear. The aim of (OLSEM) is to minimize the sum of the squared differences between the expected sample and the observed sample.²⁵

$$\begin{aligned} \sum_{i=1}^n \epsilon_i^2 &= \sum_{i=1}^n (y_i - E(y_i))^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2 \\ &= \sum_{i=1}^n [\hat{F}(X_i) - F(X_i)]^2 \end{aligned}$$

The empirical cdf is $F(X_i) = \frac{i-0.5}{n}$

$$\begin{aligned} \sum_{i=1}^n \epsilon_i^2 &= \sum_{i=1}^n \left[1 - \frac{\beta^2 + \beta x_i + 1}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} - \frac{i-0.5}{n} \right]^2 \\ &= \sum_{i=1}^n \left[\frac{-\beta x_i}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} - \frac{i-0.5}{n} \right]^2 \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{\partial \sum \epsilon^2}{\partial \beta} &= 2 \sum_{i=1}^n \left(\frac{-\beta x_i}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} - \frac{i-0.5}{n} \right) \left(\frac{\beta x_i^2}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} + \frac{\beta^2 x_i - x_i}{(\beta^2 + 1)^2} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) = K(\beta) \end{aligned} \quad (18)$$

$$\frac{\partial \sum \epsilon^2}{\partial \gamma} = 2 \sum_{i=1}^n \left(\frac{-\beta x_i}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} - \frac{i-0.5}{n} \right) \left(\frac{\beta x_i}{\beta^2 + 1} \cdot \frac{x_i^2}{\gamma^3} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) = L(\gamma) \quad (19)$$

The Newton-Raphson method is used to solve non-linear equations, and its steps are as follows

$$\begin{bmatrix} \hat{\beta}_{i+1} \\ \hat{\gamma}_{i+1} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_i \\ \hat{\gamma}_i \end{bmatrix} - j^{-1} \begin{bmatrix} K(\beta) \\ L(\gamma) \end{bmatrix}, \text{ where } j = \begin{bmatrix} \frac{\partial K(\beta)}{\partial \beta} & \frac{\partial K(\beta)}{\partial \gamma} \\ \frac{\partial L(\gamma)}{\partial \beta} & \frac{\partial L(\gamma)}{\partial \gamma} \end{bmatrix}, \text{ is a Jacobean matrix}$$

$$\begin{aligned} \frac{\partial K(\beta)}{\partial \beta} &= 2 \sum_{i=1}^n \left(\frac{-\beta x_i}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} - \frac{i-0.5}{n} \right) \left(\frac{-\beta x_i^3}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} + \frac{x_i^2 - \beta^2 x_i^2}{(\beta^2 + 1)^2} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) + \\ &\left(\frac{x_i^2 - \beta^2 x_i^2}{(\beta^2 + 1)^2} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} + \frac{2\beta x_i(3 - \beta^2)}{(\beta^2 + 1)^3} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) + \\ &\left(\frac{-\beta x_i^2}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} + \frac{x_i^2 - \beta^2 x_i^2}{(\beta^2 + 1)^2} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) \left(\frac{\beta x_i^2}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} + \right. \\ &\left. \frac{x_i^2 - \beta^2 x_i^2}{(\beta^2 + 1)^2} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) \end{aligned} \quad (20)$$

$$\frac{\partial L(\gamma)}{\partial \gamma} = 2 \sum_{i=1}^n \left(\frac{-\beta x_i}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} - \frac{i-0.5}{n} \right) \left(\frac{\beta x_i}{\beta^2 + 1} \cdot \frac{x_i^4}{\gamma^6} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) + \left(\frac{\beta x_i}{\beta^2 + 1} \cdot \frac{x_i^2}{\gamma^3} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) \quad (21)$$

$$\begin{aligned} \frac{\partial L(\gamma)}{\partial \beta} &= 2 \sum_{i=1}^n \left(\frac{-\beta x_i}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} - \frac{i-0.5}{n} \right) \left(\frac{-\beta x_i^2}{\beta^2 + 1} \cdot \frac{x_i^2}{\gamma^3} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} + \frac{\beta^2 x_i - x_i}{(\beta^2 + 1)^2} \cdot \frac{-x_i^2}{\gamma^3} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) + \\ &\left(\frac{\beta x_i}{\beta^2 + 1} \cdot \frac{x_i^2}{\gamma^3} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) \left(\frac{\beta x_i^2}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} + \frac{\beta^2 x_i - x_i}{(\beta^2 + 1)^2} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{\partial K(\beta)}{\partial \gamma} &= 2 \sum_{i=1}^n \left(\frac{-\beta x_i}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} - \frac{i-0.5}{n} \right) \left(\frac{-\beta x_i^2}{\beta^2 + 1} \cdot \frac{x_i^2}{\gamma^3} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} + \frac{\beta^2 x_i - x_i}{(\beta^2 + 1)^2} \cdot \frac{-x_i^2}{\gamma^3} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) + \\ &\left(\frac{\beta x_i}{\beta^2 + 1} \cdot \frac{x_i^2}{\gamma^3} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) \left(\frac{\beta x_i^2}{\beta^2 + 1} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} + \frac{\beta^2 x_i - x_i}{(\beta^2 + 1)^2} e^{-\left(\beta x_i + \frac{x_i^2}{2(\gamma^2)} \right)} \right) \end{aligned} \quad (23)$$

Observe that: $\frac{\partial K(\beta)}{\partial \gamma} = \frac{\partial L(\gamma)}{\partial \beta}$ since this Jacobian matrix is symmetric.

4.2 Data Study

4.2.1 Simulation Study

In this section, we produce different sizes of random samples, which rely on the CDF of the

(SHR) distribution $F(X) = 1 - \left(\frac{\beta^2 + \beta x + 1}{\beta^2 + 1} \cdot e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} \right)$ Let x be a continuous random variable with

the distribution of (SHR), then the Quantile function is

Let $F(x) = v$, $v \sim V(0,1)$, then $F(x) = V$, $x = F^{-1}(v)$

$$v = \left(1 - \frac{\beta^2 + \beta x + 1}{\beta^2 + 1} \cdot e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} \right)$$

$$v - 1 = - \left(1 - \frac{\beta^2 + \beta x + 1}{\beta^2 + 1} \cdot e^{-\left(\beta x + \frac{x^2}{2\gamma^2}\right)} \right)$$

This is a non-linear equation. Therefore, to solve it we used computer programming and numerical methods namely, mathematical concerning or MATLAB X. The different sample sizes ($n=20, 30, 50, 100, 200$), which is produced through the function. And, estimation of the parameter by means of MLE and OLS methods. So, using the MSE to contrast these method

$MSE = \sum_{i=1}^n \frac{(\theta - \hat{\theta})^2}{L}$ where n is the size of the sample, and L is the number of repeated experiments ($L=50$), and $\theta = (\beta, \gamma)$.

We can see from this simulation test that the top correct method is the method that repeats the most which is (OLS), as the greatest method for the variety of sample sizes taken, as the outcome of comparing it with different methods taken, because it has a smaller average square.

Table 1. MSE of β for the E1 experiment($\beta=1.5, \gamma=2$)

<i>L</i>	<i>n</i>	MLE	OLS	BEST
1000	20	0.2495	0.0261	OLS
	30	0.1560	0.0217	OLS
	50	0.0932	0.0169	OLS
	100	0.0443	0.0096	OLS
	200	0.0220	0.0072	OLS
2000	20	0.2313	0.0280	OLS
	30	0.1478	0.0215	OLS
	50	0.0917	0.0158	OLS
	100	0.0453	0.0106	OLS
	200	0.0215	0.0071	OLS

Table 2. MSE of γ for the E1 experiment($\beta=1.5, \gamma=2$)

<i>L</i>	<i>n</i>	MLE	OLS	BEST
100	20	1.1249	0.0886	OLS
	30	1.0108	0.0849	MLS
	50	0.6354	0.0703	OLS
	100	0.3111	0.0487	OLS
	200	0.1114	0.0289	OLS
2000	20	1.1382	0.0951	OLS
	30	0.9622	0.0864	OLS
	50	0.6839	0.0723	OLS
	100	0.3144	0.0521	OLS
	200	0.1086	0.0293	OLS

Table 3. MSE of β for the E2 experiment($\beta=0.5, \gamma = 1.25$)

L	n	MLE	OLS	BEST
1000	20	0.1429	0.0643	OLS
	30	0.1257	0.0542	MLS
	50	0.0913	0.0521	OLS
	100	0.0651	0.0475	OLS
	200	0.0421	0.0462	MLS
2000	20	0.1531	0.0574	OLS
	30	0.1228	0.0546	OLS
	50	0.0934	0.0514	OLS
	100	0.0682	0.0477	OLS
	200	0.0416	0.0477	MLS

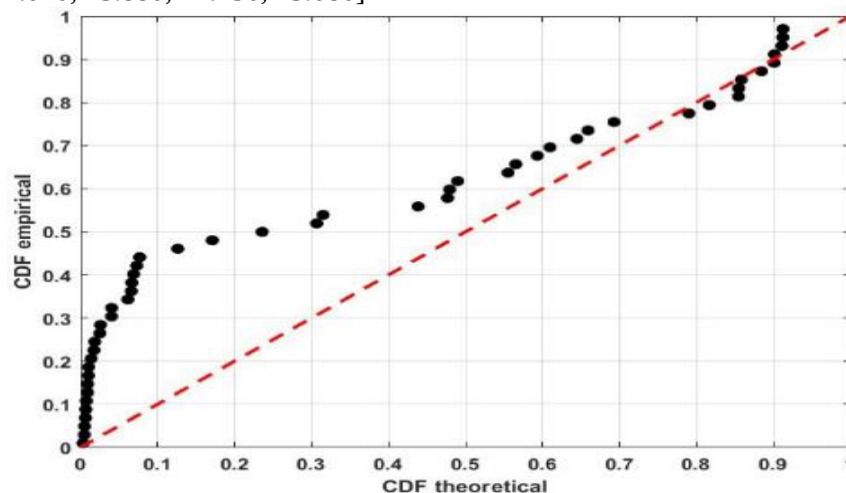
Table 4. MSE of γ for the E2 experiment($\beta=0.5, \gamma = 1.25$)

L	n	MLE	OLS	BEST
1000	20	0.0506	0.0393	OLS
	30	0.0334	0.0288	OLS
	50	0.0197	0.0258	MLS
	100	0.0092	0.0220	MLS
	200	0.0049	0.0209	MLS
2000	20	0.0569	0.0338	OLS
	30	0.0343	0.0297	OLS
	50	0.0182	0.0264	MLS
	100	0.0105	0.0227	MLS
	200	0.0050	0.0204	MLS

4.2.2 Real Data

These data were obtained by ²⁶. This applied study comprises 50 samples, used as a realistic statistical model to demonstrate the efficiency and capability of the Shanker-Rayleigh distribution in modeling data and analyzing survival or failure rates from and illustrates the failure time of 50 models (per 1000 h). The sample:

$X = [0.036, 0.058, 0.061, 0.074, 0.078, 0.086, 0.102, 0.103, 0.114, 0.116, 0.148, 0.183, 0.192, 0.254, 0.262, 0.379, 0.381, 0.538, 0.570, 0.574, 0.590, 0.618, 0.645, 0.961, 1.228, 1.600, 2.006, 2.054, 2.804, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816, 7.896, 7.904, 8.022, 9.337, 10.940, 11.020, 13.880, 14.730, 15.080]$

**Figure 4.** Plots of the probability density function (cdf) of the Shanker-Rayleigh distribution.

By applying the odds ratio test to test that this specimen has a new distribution, and as mentioned in the result, we found that this sample has a Shanker-Rayleigh distribution as follows if $\theta = (\beta = 1.5, \gamma = 7.5)$

4.2.3 Application:

Now, using estimated values for two parameters in the Shanker-Rayleigh by MLE to find numerical values for the survival function, probability density function, hazard function, and cumulative distribution function as follows:

Table 5. Estimated values for function $f(t)$, $s(t)$, $F(t)$, and $h(t)$ for MLE.

T	X(t)	f(x)	F(x)	S(X)	h(x)
1	1.600	0.181851418	0.856011475	0.143988525	1.262957719
2	0.058	0.989594145	0.058896735	0.941103265	1.051525567
3	2.006	0.106733185	0.917644613	0.082355387	1.296007325
4	3.704	0.008970894	0.993571604	0.006428396	1.395510483
5	0.078	0.972905093	0.078589081	0.921410919	1.055886221
6	0.086	0.96625648	0.086377815	0.913622185	1.05761057
7	6.816	4.35E-05	0.999971159	2.88E-05	1.509847818
8	11.020	8.70E-09	0.999999995	5.37E-09	1.620076937
9	0.103	0.952182866	0.102761996	0.897238004	1.061237779
10	0.114	0.943117955	0.113242893	0.886757107	1.063558383
11	0.148	0.915320926	0.145042064	0.854957936	1.070603461
12	7.896	5.60E-06	0.999996366	3.63E-06	1.540996097
13	0.254	0.831126033	0.238481495	0.761518505	1.091406219
14	0.381	0.736029432	0.339443925	0.660556075	1.114257306
15	0.538	0.628373407	0.448714555	0.551285445	1.139833116
16	3.058	0.024001725	0.982390061	0.017609939	1.362964674
17	0.590	0.595260121	0.481359626	0.518640374	1.147731936
18	3.931	0.006278987	0.99553379	0.00446621	1.405887287
19	0.618	0.577961385	0.498243949	0.501756051	1.151877258
20	0.645	0.561632949	0.514076688	0.485923312	1.15580573
21	0.961	0.395690841	0.669529849	0.330470151	1.197357278
22	14.730	1.50E-12	1	8.83E-13	1.702608985
23	1.228	0.288856716	0.764631174	0.235368826	1.227251378
24	2.054	0.100042836	0.923019466	0.076980534	1.299586161
25	0.262	0.824939685	0.245189198	0.754810802	1.092909219
26	3.076	0.023367727	0.982867505	0.017132495	1.36394182
27	0.183	0.887085299	0.176842015	0.823157985	1.077661051
28	3.147	0.021018189	0.984633049	0.015366951	1.367752701
29	3.625	0.010143657	0.992711751	0.007288249	1.391782574
30	0.379	0.737473456	0.337943617	0.662056383	1.11391337
31	4.073	0.005009111	0.996452833	0.003547167	1.412143917
32	0.074	0.976235329	0.074675831	0.925324169	1.05501981
33	4.393	0.002987589	0.997904404	0.002095596	1.425651007
34	4.893	0.001305458	0.999096779	0.000903221	1.445336919
35	0.061	0.987084876	0.061870698	0.938129302	1.05218425
36	6.274	0.000117406	0.999921366	7.86E-05	1.493063724
37	0.102	0.9530086	0.101804493	0.898195507	1.061025793
38	0.116	0.941473426	0.11513835	0.88486165	1.06397811
39	7.904	5.51E-06	0.999996422	3.58E-06	1.541217263
40	2.804	0.034872017	0.974143377	0.025856623	1.348668631
41	10.940	1.04E-08	0.999999994	6.41E-09	1.618191851
42	13.880	1.20E-11	1	7.12E-12	1.684429911
43	0.570	0.607844271	0.469004246	0.530995754	1.144725294
44	15.080	6.29E-13	1	3.68E-13	1.710208045
45	0.036	1.008054705	0.036870327	0.963129673	1.046644843
46	0.574	0.605312254	0.4714952	0.5285048	1.145329718
47	9.337	3.12E-07	0.999999802	1.98E-07	1.579062306
48	8.022	4.38E-06	0.999997164	2.84E-06	1.544465064
49	4.534	0.002371399	0.998343261	0.001656739	1.431365563
50	0.192	0.879891117	0.184867107	0.815132893	1.07944499

Note that:

- * Taking the real data the time to failure is increasing, giving rise to a decrease in the probability density function.
- * When the failure time increases, the cumulative distribution function $F(x)$ is increasing; this means there is a direct proportional relationship.
- * The value of the hazard function $h(t)$ is increasing with increasing failure times; that means there is a direct relationship between failure and the hazard function.

5. Conclusion

A new statistical lifetime distribution was introduced, which is a mixture of the Shanker and Rayleigh distributions depending on their survival functions and is called the "Shanker- Rayleigh distribution". In addition, the shape of the PDF, CDF, and the hazard function were discussed. In addition, the basic quantitative and mathematical characteristics of the new distribution, including the mode, quantile function, moments about the origin, characteristic function, moment generation function, factorial moment generation function, and mean time to failure. We also present two methods for classical estimation: maximum likelihood estimation and the ordinary least squares method. In addition, we conducted a simulation study and an application to real data.

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Conflict of Interest

There exists no conflict of interest.

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