



The Bifurcation and Persistence Conditions of an Eco-Toxicant-Epidemiological Model with the Impact of Michaelis-Menten Type of Harvesting of Susceptible Predators and Allee-effect of Species

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Abstract:

This paper investigates the conditions for local bifurcation in a toxicological ecology model containing a susceptible predator, an infected predator, and a prey, with strong Allee-effect and toxicity for both the susceptible predator and the prey, non-linear harvesting of the Michaelis-Menten type in the susceptible predator group, and treatment for the infected predator. The model was studied after transformation into a dimensionless System. The model possesses eleven equilibrium points (equilibria), with the saddle-node bifurcation occurring around the equilibrium points E_7 and E_8 . Additionally, transcritical and pitchfork bifurcations transpired at all other equilibrium points. The Hopf bifurcation of the positive point was studied, and the conditions for this point were determined. The persistence was studied numerically, and its points were determined. The numerical results of a set of hypothetical parameters supported our analytical results for demonstrating local bifurcation and its persistence using C++ and MATLAB software to further refine the results.

Keywords: Eco-toxicant-epidemiological, Strong-Allee-effect, Michaelis-Menten, Treatment, Persistence, Bifurcation.

1. Introduction

Mathematical modeling is a critical tool for comprehending the intricate relationships between species and environmental variables in ecosystems and epidemics. These models allow for the examination of population dynamics, the evaluation of the influence of biological and environmental variables, and the investigation of future possibilities for resource management or disease control¹⁻³. Among the core analytical techniques in this context, bifurcation analysis stands out as a useful mathematical method for investigating qualitative changes in system behavior as its parameters change. This enables the identification of transitions from stability to instability, as well as the creation of cyclical or chaotic events^{4,5}.

The interaction of biological processes (such as predation, infection, reproduction, and mortality) with environmental variables (such as toxicity, harvesting, or climatic change) makes ecosystems and epidemics very complicated. As a result, developing mathematical models that account for these interactions, including susceptible and infected predators, prey subjected to strong mechanistic influences, environmental toxicity, nonlinear harvesting, and a treatment function, is a significant scientific and practical challenge⁶⁻⁸.

Bifurcation is a qualitative change in a system's behavior that occurs when a specific parameter changes, causing the system to transition from a stable to a non-stable state, or vice versa, or to develop new cyclical solutions (boundary cycles), or to change the number of equilibrium points⁹⁻¹¹. Bifurcation is an important method for studying how ecological and epidemiological systems adapt to environmental changes or human interventions (such harvesting or treatment).

There are several varieties of bifurcation, including Saddle-node-bifurcation, Transcritical-bifurcation, Pitchfork-bifurcation, and Hopf-bifurcation, each having unique mathematical and biological properties^{12,13}.

Hopf Bifurcation is very important in environmental and epidemiological models because it is related to the creation of rhythmic activity in a system, which is seen in many environmental phenomena such as predator-prey cycles or epidemic outbreaks^{10, 14-16}. Mathematical models for analyzing predator-prey interactions have their origins in the classic Lotka-Volterra model, which was established in the 1920s and served as a foundation for understanding population dynamics in ecosystems^{17, 18}. Later, other features, including infection (epidemiological models), mechanistic impact, toxicity, harvesting, and treatment functions, were included in the models, making them more realistic and sophisticated¹⁹.

Modern eco-epidemiological models assume one prey and two predators: one healthy (uninfected) and one ill (infected with a contagious disease), with the possibility of transmission between predators or prey to predator. This mimics genuine ecological circumstances, such as disease transmission among wild predators^{20, 21}.

The Allee effect is a biological phenomenon in which population growth rates can be negative or low at low population densities due to mating problems, a lack of cooperation, increased predation, or other causes^{1, 22,23}.

The Allee effect is characterized as two types:

- Strong Allee effect: There is a crucial threshold for population density; if it goes below this level, the species is extinct.
- Weak Allee effect: At low densities, growth rates decline but remain positive.

Moreover, environmental toxicity has a significant impact on population dynamics, including higher mortality rates, lower fertility, or changed biological behavior in organisms²⁴. Toxicity can come from either natural (toxic algae) or anthropogenic sources (chemical pollution).

Harvesting is the process of eliminating a subset of a population (prey or predator) for human use (hunting, poaching, or environmental management). In fact, harvesting is frequently nonlinear, growing with density until it hits saturation due to effort, human resources, or technological constraints^{22, 25, 26}. Nonlinear harvesting (Michaelis-Menten) causes the creation of novel bifurcation patterns (such as Hopf bifurcation or nodal bifurcation) and can stabilize or destabilize the system depending on the harvest intensity and parameters^{12,25,27}.

Furthermore, the interplay of harvesting with toxicity or the impacts of automation might result in unanticipated consequences such as rapid extinction or the establishment of borderline cycles. Treatment is a biological or medical intervention that aims to reduce the number of infected people or increase recovery rates, and it plays an important role in the dynamics of epidemiological and environmental systems. Incorporating a treatment function changes the system's equilibrium and stability, potentially reducing the disease's basic reproduction number (R_0), eradicating the disease at treatment-critical values, or introducing new bifurcation patterns like reverse or Hopf bifurcation²⁸⁻³¹.

Recently,³² proposed and examined an epidemiological ecology model incorporating logistic growth; fear; a strong Allee effect; toxicity for both healthy predators and prey; and nonlinear harvesting of the type of Michael-Menten function in the susceptible predator population, along with competition for logistic growth in the infected prey population with the susceptible population, treatment in the infected predator population, and the susceptible predator being fed on prey only.

The goal of this study is to define the requirements for local bifurcation at equilibrium points and Hopf-bifurcation located near positive equilibrium points for the mathematical model presented in³², as well as to validate the analytical results using numerical simulation with MATLAB.

2. Mathematical Model:

The following is an eco-toxicant-epidemiological model that has been developed and presented in ³²: it takes into consideration the Michaelis-Menten type of harvesting of vulnerable predators and the treatment of infected predators, and the following is how the model is structured:

$$\begin{aligned}\frac{dZ_1}{dT} &= r_1 Z_1 \left(1 - \frac{Z_1}{k_1}\right) (Z_1 - A_1) - \frac{a Z_1 Z_S}{1 + ab Z_1} - \gamma_1 Z_1^2 - d_1 Z_1, \\ \frac{dZ_S}{dT} &= r_2 Z_S \left(1 - \frac{Z_S + Z_I}{k_2}\right) (Z_S - A_2) + \frac{c_1 a Z_1 Z_S}{1 + ab Z_1} - \gamma_2 Z_S - d_2 Z_S - \frac{Eq Z_S}{n_1 E + n_2 Z_S} - \beta 3.05 Z_S Z_I + \frac{\rho Z_I}{\sigma + Z_I}, \\ \frac{dZ_I}{dT} &= \beta Z_S Z_I - d_3 Z_I - \frac{\rho Z_I}{\sigma + Z_I}.\end{aligned}\quad (1)$$

With the initial condition. $Z_1(0) \geq 0$, $Z_S(0) \geq 0$ and $Z_I(0) \geq 0$. As can be observed from **Table 1**, the following system characteristics have biological significance:

Table 1. Biological significance of System 1 parameters.

Parameters	Biological meaning
$r_i > 0, i = 1, 2$	The inherent growth rate of the prey population and vulnerable predator population
$K_i > 0, i = 1, 2$	The carrying capacity of the prey population and the vulnerable predator population
$A_i > 0, i = 1, 2$	Allee's effect on the prey population and the sensitive predator population
$a > 0$	Predator and prey attack rates, correspondingly
$b > 0$	The handling time of the prey
$\gamma_1 > 0$	The toxicant environment rate on the prey
$c_1 \in (0, 1)$	Food conversion rate to susceptible predators
$\gamma_2 > 0$	The toxicant environment rate on the susceptible predator
$q > 0$	The catchability coefficient
$E > 0$	The external effort devoted to harvesting
$n_i > 0, i = 1, 2$	The constants
$\beta > 0$	The infection rate
$\rho > 0$	The treatment rate
$\sigma > 0$	The depletion rate of the treatment
$d_i > 0, i = 1, 2, 3$	Death due to external sources rates of prey, susceptible predators, and infected predators, respectively.

Subsequently, the following dimensionless **System 1** is derived:

$$\begin{aligned}\frac{dx}{dt} &= x \left[(1 - x)(x - \tilde{n}_1) - \frac{y_S}{\tilde{n}_2 + x} - \tilde{n}_3 x - \tilde{n}_4 \right], \\ \frac{dy_S}{dt} &= \tilde{n}_5 y_S (1 - \tilde{n}_6 (y_S + y_I)) (y_S - \tilde{n}_7) + \frac{\tilde{n}_8 x y_S}{\tilde{n}_2 + x} - \tilde{n}_9 y_S - \tilde{n}_{10} y_S - \frac{\tilde{n}_{11} y_S}{\tilde{n}_{12} + y_S} - \tilde{n}_{13} y_I y_S + \frac{\tilde{n}_{14} y_I}{\tilde{n}_{15} + y_I}, \\ \frac{dy_I}{dt} &= y_I \left[\tilde{n}_{13} y_S - \tilde{n}_{16} - \frac{\tilde{n}_{14}}{\tilde{n}_{15} + y_I} \right].\end{aligned}\quad (2)$$

$$\text{where } x = \frac{Z_1}{k_1}, y_S = \frac{Z_S}{br_1 k_1^2}, y_I = \frac{Z_I}{br_1 k_1^2}, t = r_1 k_1 T$$

$$\begin{aligned}\tilde{n}_1 &= \frac{A_1}{k_1}, \tilde{n}_2 = \frac{1}{abk_1}, \tilde{n}_3 = \frac{\gamma_1}{r_1}, \tilde{n}_4 = \frac{d_1}{r_1 k_1}, \tilde{n}_5 = r_1 b k_1, \tilde{n}_6 = \frac{br_1 k_1^2}{k_2}, \tilde{n}_7 = \frac{A_2}{br_1 k_1^2}, \tilde{n}_8 = \frac{c_1}{r_1 b k_1}, \\ \tilde{n}_9 &= \frac{\gamma_2}{r_1 k_1}, \tilde{n}_{10} = \frac{d_2}{r_1 k_1}, \tilde{n}_{11} = \frac{Eq}{n_2 b r_1^2 K_1^3}, \tilde{n}_{12} = \frac{n_1 E}{n_2 b r_1 K_1^2}, \tilde{n}_{13} = B b k, \tilde{n}_{14} = \frac{\rho}{r_1^2 b k_1^3}, \tilde{n}_{15} = \frac{\sigma}{br_1 k_1^2}, \\ \tilde{n}_{16} &= \frac{d_3}{r_1 k_1}.\end{aligned}$$

2.1. Local Bifurcation Analysis

This section examines the dynamic behavior at each equilibrium point (EP) of the **System 2** as a function of modifying certain parameters. Remember that the presence of the **System's 2** non-hyperbolic (EP) is necessary for BIF to occur. Hence, suitable applications of Sotomayor's theorem include the following theorems³³. Following the information in ³² on the Jacobian matrix (JM) $J(x, y_S, y_I)$ of **System 2**:

$$J = [Z_{ij}]_{3 \times 3} = \begin{bmatrix} Z_{11} & Z_{12} & 0 \\ Z_{21} & Z_{22} & Z_{23} \\ 0 & Z_{32} & Z_{33} \end{bmatrix}, \quad (3)$$

where $Z_{11} = -3x^2 - 2(\delta_2)x - \frac{\tilde{n}_2 y_S}{(\tilde{n}_2 + x)^2} - (\delta_1)$, $Z_{12} = \frac{-x}{\tilde{n}_2 + x}$, $Z_{21} = \frac{\tilde{n}_2 \tilde{n}_8 y_S}{(\tilde{n}_2 + x)^2}$,
 $Z_{22} = -3\tilde{n}_5 \tilde{n}_6 y_S^2 + 2\tilde{n}_5(1 - \tilde{n}_6 y_I + \tilde{n}_6 \tilde{n}_7) y_S + (\varepsilon_2) y_I + \frac{\tilde{n}_8 x}{\tilde{n}_2 + x} - \frac{\tilde{n}_{11} \tilde{n}_{12}}{(\tilde{n}_{12} + y_S)^2} - (\delta_4)$,
 $Z_{23} = -\tilde{n}_5 \tilde{n}_6 y_S^2 + (\varepsilon_2) y_S + \frac{\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I)^2}$, $Z_{32} = \tilde{n}_{13} y_I$, $Z_{33} = \tilde{n}_{13} y_S - \tilde{n}_{16} - \frac{\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I)^2}$,
 $\varepsilon_2 = \tilde{n}_5 \tilde{n}_6 \tilde{n}_7 - \tilde{n}_{13}$, $\delta_1 = (\tilde{n}_1 + \tilde{n}_4)$, $\delta_2 = \tilde{n}_3 - (\vartheta_1)$, $\vartheta_1 = 1 + \tilde{n}_1$, $\delta_3 = (\tilde{n}_9 + \tilde{n}_{10})$,
 $\delta_4 = \delta_3 + \tilde{n}_5 \tilde{n}_7$, $\alpha_4 = 1 + \tilde{n}_6 \tilde{n}_7$.

It is evident that for every non-zero vector $\check{v} = (\check{v}_1, \check{v}_2, \check{v}_3)^T$, the following applies:

$$D^2 F(X, \mu)(\check{v}, \check{v}) = [\check{H}_{11}]_{3 \times 1}, \quad (4)$$

$$\check{H}_{11} = -2[(-3x + (1 - (\tilde{n}_1 + \tilde{n}_3))) + \frac{\tilde{n}_2 y_S}{(\tilde{n}_2 + x)^3}] \check{v}_1^2 - \frac{\tilde{n}_2}{(\tilde{n}_2 + x)^2} \check{v}_2 \check{v}_1],$$

$$\check{H}_{21} = -\frac{2\tilde{n}_2 \tilde{n}_8 y_S}{(\tilde{n}_2 + x)^3} \check{v}_1^2 + \frac{\tilde{n}_8}{(\tilde{n}_2 + x)^2} (\tilde{n}_2 + 1) \check{v}_2 \check{v}_1 + 2\tilde{n}_5 (1 + \tilde{n}_6 (\tilde{n}_7 - (3 + y_I))) \check{v}_2^2$$

$$- [2\tilde{n}_5 \tilde{n}_6 (2y_S - \tilde{n}_7) + \tilde{n}_{13}] \check{v}_3 \check{v}_2 - \frac{2\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I)^3} \check{v}_3^2,$$

$$\check{H}_{31} = 2[\tilde{n}_{13} \check{v}_3 \check{v}_2 + \frac{\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I)^3} \check{v}_3^2].$$

$$\text{and } D^3 F(X, \mu)(\check{v}, \check{v}, \check{v}) = [\mathbb{Q}_{11}]_{3 \times 1}, \quad (5)$$

$$\mathbb{Q}_{11} = -6[\check{v}_1 + \frac{\tilde{n}_2}{(\tilde{n}_2 + x)^3} (\frac{1}{(\tilde{n}_2 + x)} y_S \check{v}_1 - \check{v}_2)] \check{v}_1^2,$$

$$\mathbb{Q}_{21} = -2[\frac{3\tilde{n}_2 \tilde{n}_8 y_S}{(\tilde{n}_2 + x)^4} \check{v}_1^3 + \frac{\tilde{n}_8}{(\tilde{n}_2 + x)^3} (2\tilde{n}_2 + 1) \check{v}_2 \check{v}_1^2 + 3\tilde{n}_5 \tilde{n}_6 (\check{v}_2 + \check{v}_3) \check{v}_2^2 - \frac{3\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I)^4} \check{v}_3^3],$$

$$\mathbb{Q}_{31} = \frac{-6\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I)^4} \check{v}_3^3.$$

Where $X = (x, y_S, y_I)^T$ and μ is any parameter.

2.1.1. Theorem: Pretend that the following condition is true and that conditions (4 and 11) are stated in ³²:

$$\frac{\tilde{n}_8 \tilde{x}}{\tilde{n}_2 + \tilde{x}} > \delta_4, \quad (6)$$

$$\check{\chi}_3 \neq \check{\chi}_4, \quad (7)$$

$$\text{where } \check{\chi}_3 = \frac{-\tilde{n}_8}{(\tilde{n}_2 + x)^2} (\tilde{n}_2 + 1) \mathfrak{X} + 6\tilde{n}_5 \tilde{n}_6, \quad \check{\chi}_4 = 2\tilde{n}_5 (1 + \tilde{n}_6 \tilde{n}_7).$$

At the (EP) $E_1 = (\tilde{x}, 0, 0)$ in **System 2**, with the parameter $\tilde{n}_{11} = \tilde{n}_{12} (\frac{\tilde{n}_8 \tilde{x}}{\tilde{n}_2 + \tilde{x}} - \delta_4)$, we can see Transcritical and Pitchfork bifurcation (TB and PB), however saddle-node bifurcation (SNB) is not possible.

Proof: Based on the Jacobian matrix J_1 , which is provided in **Equation 10** in reference³², **System 2** at the (EP) E_1 has an eigenvalue of zero at $\tilde{n}_{11} = \tilde{n}_{11} = \tilde{n}_{12} (\frac{\tilde{n}_8 \tilde{x}}{\tilde{n}_2 + \tilde{x}} - \delta_4)$, and J_1 with $\tilde{n}_{11} = \tilde{n}_{11}$ becomes

$$\tilde{J}_1 = J(E_1, \tilde{n}_{11}) = \begin{bmatrix} -(3\tilde{x}^2 + 2(\delta_2)\tilde{x} + (\delta_1)) & -\frac{\tilde{x}}{\tilde{n}_2 + \tilde{x}} & 0 \\ 0 & 0 & \frac{\tilde{n}_{14}}{\tilde{n}_{15}} \\ 0 & 0 & -(\tilde{n}_{16} + \frac{\tilde{n}_{14}}{\tilde{n}_{15}}) \end{bmatrix}$$

First, let $\check{v}^{[1]} = (\check{v}_1^{[1]}, \check{v}_2^{[1]}, \check{v}_3^{[1]})^T$ serve as the eigenvector that is associated with the eigenvalue $\lambda_{1y_S} = 0$ for $\tilde{J}_1$. Thus $(\tilde{J}_1 - \lambda_{1y_S} I)\check{v}^{[1]} = 0$, that gives $\check{v}^{[1]} = (\aleph \check{v}_2^{[1]}, \check{v}_2^{[1]}, 0)^T$, where $\aleph = \frac{-\tilde{x}}{(\tilde{n}_2 + \tilde{x})[3\tilde{x}^2 + 2(\delta_2)\tilde{x} + (\delta_1)]}$, $\check{v}_2^{[1]} \neq 0$ the expression is valid.

Let $\mathbb{W}^{[1]} = (\mathbb{W}_1^{[1]}, \mathbb{W}_2^{[1]}, \mathbb{W}_3^{[1]})^T$ represents the eigenvector of $\tilde{J}_1^T$ for $\lambda_{1y_S} = 0$ for $\tilde{J}_1$.

Then we get $(\tilde{J}_1^T - \lambda_{1y_S} I)\mathbb{W}^{[1]} = 0$ then by solving this equation for $\mathbb{W}^{[1]}$ we get $\mathbb{W}^{[1]} = (0, \mathbb{W}_2^{[1]}, \mathfrak{U}\mathbb{W}_2^{[1]})^T$, where $\mathfrak{U} = \frac{\tilde{n}_{14}}{\tilde{n}_{15}(\tilde{n}_{16} + \frac{\tilde{n}_{14}}{\tilde{n}_{15}})}$, $\mathbb{W}_2^{[1]} \neq 0$ the expression is valid.

Second, consider $\frac{\partial f}{\partial \tilde{n}_{11}} = f_{\tilde{n}_{11}}(X, \tilde{n}_{11}) = \left(\frac{\partial f_1}{\partial \tilde{n}_{11}}, \frac{\partial f_2}{\partial \tilde{n}_{11}}, \frac{\partial f_3}{\partial \tilde{n}_{11}}\right)^T = \left(0, \frac{y_S}{\tilde{n}_{12} + y_S}, 0\right)^T$.

So, $f_{\tilde{n}_{11}}(E_1, \tilde{n}_{11}) = (0, 0, 0)^T$ and hence $(\mathbb{W}^{[1]})^T f_{\tilde{n}_{11}}(E_1, \tilde{n}_{11}) = 0$.

Therefore, using "Sotomayor's theorem" (SNB)'s conditions cannot be satisfied. While the (TB)'s first condition is satisfied.

Now, given that $Df_{\tilde{n}_{11}}(X, \tilde{n}_{11}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\tilde{n}_{12}}{(\tilde{n}_{12} + y_S)^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where $Df_{\tilde{n}_{11}}(X, \tilde{n}_{11})$ is the derivative of $f_{\tilde{n}_{11}}(X, \tilde{n}_{11})$ with respect to $X = (x, y_S, y_I)^T$.

Further, it is observed that $Df_{\tilde{n}_{11}}(E_1, \tilde{n}_{11})\check{v}^{[1]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\tilde{n}_{12}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \aleph \check{v}_2^{[1]} \\ \check{v}_2^{[1]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\tilde{n}_{12}} \check{v}_2^{[1]} \\ 0 \end{bmatrix}$, so

$$(\mathbb{W}^{[1]})^T [Df_{\tilde{n}_{11}}(E_1, \tilde{n}_{11})\check{v}^{[1]}] = (0, \mathbb{W}_2^{[1]}, \mathfrak{U}\mathbb{W}_2^{[1]}) \left(0, \frac{1}{\tilde{n}_{12}} \check{v}_2^{[1]}, 0\right)^T = \frac{1}{\tilde{n}_{12}} \check{v}_2^{[1]} \mathbb{W}_2^{[1]} \neq 0$$

Moreover, by substituting $\check{v}^{[1]}$ in **Equation 4** we get

$$(\mathbb{W}^{[1]})^T [D^2 f_{\tilde{n}_{11}}(E_1, \tilde{n}_{11})(\check{v}^{[1]}, \check{v}^{[1]})] = \left[\frac{\tilde{n}_8}{(\tilde{n}_2 + \tilde{x})^2} (\tilde{n}_2 + 1)\aleph + 2\tilde{n}_5(1 + \tilde{n}_6(\tilde{n}_7 - 3)) \right] \check{v}_2^{[1]^2} \mathbb{W}_2^{[1]} \neq 0$$

regarding conditions (6 and 7), as well as conditions (4 and 11) outlined in ³². Hence, **System 2** utilizes "Sotomayor's theorem" to determine that there is a parameter $\tilde{n}_{11} = \tilde{n}_{11}$ with a TB at E_1 but there is no (PB).

While if condition (7) not hold then there is no a (TB) and by substituting $\check{v}^{[1]}$ in **Equation 5** we get

$$(\mathbb{W}^{[1]})^T [D^3 f_{\tilde{n}_{11}}(E_1, \tilde{n}_{11})(\check{v}^{[1]}, \check{v}^{[1]}, \check{v}^{[1]})] = -2 \left[\frac{\tilde{n}_8}{(\tilde{n}_2 + \tilde{x})^3} (2\tilde{n}_2 + 1)\aleph^2 + 3\tilde{n}_5\tilde{n}_6 \right] \check{v}_2^{[1]^3} \mathbb{W}_2^{[1]} \neq 0.$$

Once again, while applying "Sotomayor's theorem" to **System 2** at E_1 , we get (PB) with the parameter $\tilde{n}_{11} = \tilde{n}_{11}$. For the (EP) E_2 , the same holds.

2.1.2. Theorem: If condition (14), as stated in ³², is true, then the following condition is true:

$$\tilde{n}_{13}\hat{y}_S > \frac{\tilde{n}_{14}}{\tilde{n}_{15}}, \quad (8)$$

$$3\tilde{n}_5\tilde{n}_6\hat{y}_S^2 + \frac{\tilde{n}_{11}}{\tilde{n}_{12}} + (\delta_4) > 2\tilde{n}_5(\alpha_4)\hat{y}_S, \quad (9)$$

$$\tilde{n}_5\tilde{n}_6\hat{y}_S^2 + \tilde{n}_{13}\hat{y}_S < \frac{\tilde{n}_{14}}{\tilde{n}_{15}} + \tilde{n}_5\tilde{n}_6\tilde{n}_7\hat{y}_S, \quad (10)$$

$$\tilde{n}_{13}\mathfrak{d} + \frac{\tilde{n}_{14}}{\tilde{n}_{15}^2} \neq 0. \quad (11)$$

where $\bar{d} = \frac{-(-\tilde{n}_5\tilde{n}_6\hat{y}_S^2 + \varepsilon_2\hat{y}_S + \frac{\tilde{n}_{14}}{\tilde{n}_{15}})}{-3\tilde{n}_5\tilde{n}_6\hat{y}_S^2 + 2\tilde{n}_5(\alpha_4)\hat{y}_S - \frac{\tilde{n}_{11}}{\tilde{n}_{12}} - (\delta_4)}$.

System 2 at the (EP) $E_3 = (0, \hat{y}_S, 0)$ with the parameter $\tilde{n}_{16} = \tilde{n}_{13}\hat{y}_S - \frac{\tilde{n}_{14}}{\tilde{n}_{15}}$ has a (TB and PB) but (SNB) cannot occur.

Proof: From the Jacobian matrix J_3 given in **Equation 13** as in ³² the **System 2** at the (EP) E_3 has eigenvalue ($\lambda_{3y_1} = 0$) equal to zero at $\hat{n}_{16} = \tilde{n}_{16} = \tilde{n}_{13}\hat{y}_S - \frac{\tilde{n}_{14}}{\tilde{n}_{15}}$ then J_3 with $\hat{n}_{16} = \tilde{n}_{16}$ becomes

$$\hat{J}_3 = \hat{J}(E_3, \tilde{n}_{16}) = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } c_{11} = -\frac{\hat{y}_S}{\tilde{n}_2} - (\delta_1), c_{21} = \frac{\tilde{n}_8\hat{y}_S}{\tilde{n}_2}, c_{22} = -3\tilde{n}_5\tilde{n}_6\hat{y}_S^2 + 2\tilde{n}_5(\alpha_4)\hat{y}_S - \frac{\tilde{n}_{11}}{\tilde{n}_{12}} - (\delta_4), c_{23} = -\tilde{n}_5\tilde{n}_6\hat{y}_S^2 + \varepsilon_2\hat{y}_S + \frac{\tilde{n}_{14}}{\tilde{n}_{15}}.$$

Now, let $\check{v}^{[3]} = (\check{v}_1^{[3]}, \check{v}_2^{[3]}, \check{v}_3^{[3]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{3y_1} = 0$ for $\hat{J}_3$. Thus $(\hat{J}_3 - \lambda_{3y_1}I)\check{v}^{[3]} = 0$, that gives $\check{v}^{[3]} = (0, \bar{d}\check{v}_3^{[3]}, \check{v}_3^{[3]})^T$ where $\check{v}_3^{[3]} \neq 0$ is any real number. Let $\mathbb{W}^{[3]} = (\mathbb{W}_1^{[3]}, \mathbb{W}_2^{[3]}, \mathbb{W}_3^{[3]})^T$ be the $\hat{J}_3^T$ eigenvector for $\lambda_{3y_1} = 0$. Next, we get $(\hat{J}_3^T - \lambda_{3y_1}I)\mathbb{W}^{[3]} = 0$ then by solving this equation for $\mathbb{W}^{[3]}$ we get $\mathbb{W}^{[3]} = (0, 0, \mathbb{W}_3^{[3]})^T$, where $\mathbb{W}_3^{[3]} \neq 0$ is any real number. Now, consider $\frac{\partial f}{\partial \tilde{n}_{16}} = f_{\tilde{n}_{16}}(X, \tilde{n}_{16}) = (\frac{\partial f_1}{\partial \tilde{n}_{16}}, \frac{\partial f_2}{\partial \tilde{n}_{16}}, \frac{\partial f_3}{\partial \tilde{n}_{16}})^T = (0, 0, -y_1)^T$. So, $f_{\tilde{n}_{16}}(E_3, \hat{n}_{16}) = (0, 0, 0)^T$ and hence $(\mathbb{W}^{[3]})^T f_{\tilde{n}_{16}}(E_3, \hat{n}_{16}) = 0$. Therefore, using "Sotomayor's theorem" (SNB)'s conditions can not satisfy. While the (TB)'s first condition is satisfied.

Now, since $Df_{\tilde{n}_{16}}(X, \tilde{n}_{16}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ where $Df_{\tilde{n}_{16}}(X, \tilde{n}_{16})$ is derived from $f_{\tilde{n}_{16}}(X, \tilde{n}_{16})$ with respect to $X = (x, y_S, y_1)^T$.

$$\text{Further, it is observed that } Df_{\tilde{n}_{16}}(E_3, \hat{n}_{16})\check{v}^{[3]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \bar{d}\check{v}_3^{[3]} \\ \check{v}_3^{[3]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\check{v}_3^{[3]} \end{bmatrix}, \text{ so}$$

$$(\mathbb{W}^{[3]})^T [Df_{\tilde{n}_{16}}(E_3, \hat{n}_{16})\check{v}^{[3]}] = (0, 0, \mathbb{W}_3^{[3]}) (0, 0, -\check{v}_3^{[3]})^T = -\check{v}_3^{[3]} \mathbb{W}_3^{[3]} \neq 0.$$

Moreover, by substituting $\check{v}^{[3]}$ in **Equation 4**, we get

$$(\mathbb{W}^{[3]})^T [D^2 f_{\tilde{n}_{16}}(E_3, \hat{n}_{16})(\check{v}^{[3]}, \check{v}^{[3]})] = 2[\tilde{n}_{13}\bar{d} + \frac{\tilde{n}_{14}}{\tilde{n}_{15}^2}]\check{v}_3^{[3]^2} \mathbb{W}_3^{[3]} \neq 0.$$

under condition (14) which is given in ³² and conditions (8,9,10 and 11). Therefore, by using "Sotomayor's theorem" **System 2** has a (TB) at E_3 with the parameter $\hat{n}_{16} = \tilde{n}_{16}$ but there is no (PB). While if condition (11) not hold then there is no a (TB) and by substituting $\check{v}^{[3]}$ in **Equation 5** we get $(\mathbb{W}^{[3]})^T [D^3 f_{\tilde{n}_{16}}(E_3, \hat{n}_{16})(\check{v}^{[3]}, \check{v}^{[3]}, \check{v}^{[3]})] = -6 \frac{\tilde{n}_{14}}{\tilde{n}_{15}^3} \check{v}_3^{[3]^3} \mathbb{W}_3^{[3]} \neq 0.$

Once again, **System 2** at E_3 has (PB) with the parameter $\hat{n}_{16} = \tilde{n}_{16}$ thanks to "Sotomayor's theorem.". Likewise, the (EP) E_4 .

2.1.3. Theorem: The following condition is assumed to hold if condition (4), as stated in ³², is true:

$$\tilde{n}_{13}\bar{y}_S > \frac{\tilde{n}_{14}}{\tilde{n}_{15}}, \quad (12)$$

$$3\bar{x}^2 + \frac{\tilde{n}_2\bar{y}_S}{(\tilde{n}_2+\bar{x})^2} + (\delta_1) > -2(\delta_2)\bar{x}, \quad (13)$$

$$3\tilde{n}_5\tilde{n}_6\bar{y}_S^2 + \frac{\tilde{n}_{11}\tilde{n}_{12}}{(\tilde{n}_{12}+\bar{y}_S)^2} + (\delta_4) > 2\tilde{n}_5(\alpha_4)\bar{y}_S + \frac{\tilde{n}_8\bar{x}}{\tilde{n}_2+\bar{x}}, \quad (14)$$

$$\tilde{n}_5\tilde{n}_6\bar{y}_S^2 + \tilde{n}_{13}\bar{y}_S < \frac{\tilde{n}_{14}}{\tilde{n}_{15}} + \tilde{n}_5\tilde{n}_6\tilde{n}_7\bar{y}_S, \quad (15)$$

$$\tilde{n}_{13}\frac{\bar{y}_2}{\bar{y}_3} + \frac{\tilde{n}_{14}}{\tilde{n}_{15}^2} \neq 0. \quad (16)$$

Where $\bar{y}_2 = -D_{23}D_{11}, \bar{y}_3 = D_{22}D_{11} - D_{21}D_{12}$,

$$D_{11} = -3\bar{x}^2 - 2(\delta_2)\bar{x} - \frac{\tilde{n}_2\bar{y}_S}{(\tilde{n}_2+\bar{x})^2} - (\delta_1), D_{12} = \frac{-\bar{x}}{\tilde{n}_2+\bar{x}}, D_{21} = \frac{\tilde{n}_2\tilde{n}_8\bar{y}_S}{(\tilde{n}_2+\bar{x})^2}, D_{22} = -3\tilde{n}_5\tilde{n}_6\bar{y}_S^2 + 2\tilde{n}_5(\alpha_4)\bar{y}_S + \frac{\tilde{n}_8\bar{x}}{\tilde{n}_2+\bar{x}} - \frac{\tilde{n}_{11}\tilde{n}_{12}}{(\tilde{n}_{12}+\bar{y}_S)^2} - (\delta_4), D_{23} = -\tilde{n}_5\tilde{n}_6\bar{y}_S^2 + (\varepsilon_2)\bar{y}_S + \frac{\tilde{n}_{14}}{\tilde{n}_{15}}$$

System 2 at the (EP) $E_5 = (\bar{x}, \bar{y}_S, 0)$ with the parameter $\tilde{n}_{16} = \left(\tilde{n}_{13}\bar{y}_S - \frac{\tilde{n}_{14}}{\tilde{n}_{15}}\right)$ has a (TB and PB) but (SNB) can't occur.

Proof: From the Jacobian matrix J_5 given in **Equation 16** as in ³² the **System 2** at the (EP) E_5 has eigenvalue $(\lambda_{5y_1} = 0)$ equal to zero at $\tilde{n}_{16} = \tilde{n}_{16} = \left(\tilde{n}_{13}\bar{y}_S - \frac{\tilde{n}_{14}}{\tilde{n}_{15}}\right)$ then J_5 with $\tilde{n}_{16} = \tilde{n}_{16}$

$$\text{Becomes } \bar{J}_5 = \bar{J}(E_5, \tilde{n}_{16}) = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & D_{23} \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } D_{33} = \tilde{n}_{13}\bar{y}_S - \tilde{n}_{16} - \frac{\tilde{n}_{14}}{\tilde{n}_{15}}.$$

Now, let $\check{v}^{[5]} = (\check{v}_1^{[5]}, \check{v}_2^{[5]}, \check{v}_3^{[5]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{5y_1} = 0$ for $\bar{J}_5$. Thus $(\bar{J}_5 - \lambda_{5y_1}I)\check{v}^{[5]} = 0$, that gives $\check{v}^{[5]} = \left(\frac{\bar{y}_1}{\bar{y}_3}\check{v}_3^{[5]}, \frac{\bar{y}_2}{\bar{y}_3}\check{v}_3^{[5]}, \check{v}_3^{[5]}\right)^T$ where $\bar{y}_1 = D_{12}D_{23}$ and $\check{v}_3^{[5]} \neq 0$ is any real number. Let $\check{w}^{[5]} = (\check{w}_1^{[5]}, \check{w}_2^{[5]}, \check{w}_3^{[5]})^T$ be its eigenvector $\hat{J}_3^T$ for $\lambda_{5y_1} = 0$. After that, we get $(\hat{J}_3^T - \lambda_{5y_1}I)\check{w}^{[3]} = 0$ then by solving this equation for $\check{w}^{[3]}$ we get $\check{w}^{[3]} = (0, 0, \check{w}_3^{[5]})^T$, where $\check{w}_3^{[5]} \neq 0$ is any real number.

Now, consider $\frac{\partial f}{\partial \tilde{n}_{16}} = f_{\tilde{n}_{16}}(X, \tilde{n}_{16}) = \left(\frac{\partial f_1}{\partial \tilde{n}_{16}}, \frac{\partial f_2}{\partial \tilde{n}_{16}}, \frac{\partial f_3}{\partial \tilde{n}_{16}}\right)^T = (0, 0, -y_1)^T$.

So, $f_{\tilde{n}_{16}}(E_5, \tilde{n}_{16}) = (0, 0, 0)^T$ and hence $(\check{w}^{[5]})^T f_{\tilde{n}_{16}}(E_5, \tilde{n}_{16}) = 0$. Therefore, using "Sotomayor's theorem" (SNB)'s conditions can not satisfy. While the (TB)'s first condition is satisfied.

Now, since $Df_{\tilde{n}_{16}}(X, \tilde{n}_{16}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ where $Df_{\tilde{n}_{16}}(X, \tilde{n}_{16})$ is the derivative of $f_{\tilde{n}_{16}}(X, \tilde{n}_{16})$ with respect to $X = (x, y_S, y_I)^T$.

Further, it is observed that $Df_{\tilde{n}_{16}}(E_5, \tilde{n}_{16})\check{v}^{[5]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\bar{y}_1}{\bar{y}_3}\check{v}_3^{[5]} \\ \frac{\bar{y}_2}{\bar{y}_3}\check{v}_3^{[5]} \\ \check{v}_3^{[5]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\check{v}_3^{[5]} \end{bmatrix}$, so

$$(\check{w}^{[5]})^T [Df_{\tilde{n}_{16}}(E_5, \tilde{n}_{16})\check{v}^{[5]}] = (0, 0, \check{w}_3^{[5]})^T (0, 0, -\check{v}_3^{[5]})^T = -\check{v}_3^{[5]} \check{w}_3^{[5]} \neq 0$$

Moreover, by substituting $\check{v}^{[5]}$ in **Equation 4** we get

$$(\check{w}^{[5]})^T [D^2 f_{\tilde{n}_{16}}(E_5, \tilde{n}_{16})(\check{v}^{[5]}, \check{v}^{[5]})] = 2 \left[\tilde{n}_{13}\frac{\bar{y}_2}{\bar{y}_3} + \frac{\tilde{n}_{14}}{\tilde{n}_{15}^2} \right] \check{v}_3^{[5]^2} \check{w}_3^{[5]} \neq 0$$

under condition (4) which is given in ³² and conditions (12,13,14,15 and 16). Therefore, by using "Sotomayor's theorem" **System 2** has a (TB) at E_5 with the parameter $\tilde{n}_{16} = \tilde{n}_{16}$ but there

is no PB. While if the condition (16) not hold, then there is no TB, and by substituting $\check{v}^{[5]}$ in **Equation 5**, we get $(\mathbb{W}^{[5]})^T [D^3 f_{\tilde{n}_{16}}(E_5, \tilde{n}_{16})(\check{v}^{[5]}, \check{v}^{[5]}, \check{v}^{[5]})] = -6 \frac{\tilde{n}_{14}}{\tilde{n}_{15}^3} \check{v}_3^{[5]^3} \mathbb{W}_3^{[5]} \neq 0$.

Another time, the **System 2** at E_5 has (PB) with the parameter $\tilde{n}_{16} = \tilde{n}_{16}$ by using "Sotomayor's theorem.". Likewise, with the (EP) E_6 .

2.1.4. Theorem: If condition (21), as stated in ³², is true, then the following condition is true:

$$3\tilde{n}_5\tilde{n}_6\bar{y}_S^2 + 2\tilde{n}_5\tilde{n}_6\bar{y}_I\bar{y}_S + \tilde{n}_{13}\bar{y}_I + \frac{\tilde{n}_{11}\tilde{n}_{12}}{(\tilde{n}_{12} + \bar{y}_S)^2} + (\delta_4) < 2\tilde{n}_5(\alpha_4)\bar{y}_S + (\tilde{n}_5\tilde{n}_6\tilde{n}_7)\bar{y}_I - \frac{\tau_{23}\tau_{32}}{\tau_{33}}, \quad (17)$$

$$\tilde{n}_{13} < \tilde{n}_5\tilde{n}_6\tilde{n}_7, \quad (18)$$

$$\rho_1 \neq \rho_2 \quad (19)$$

where $\rho_1 = 2\tilde{n}_5(1 + \tilde{n}_6\tilde{n}_7) + 2\tilde{n}_5\tilde{n}_6\tilde{n}_7\varphi_2 + 2\left[\tilde{n}_{13}\varphi_2 + \frac{\tilde{n}_{14}\tilde{n}_{15}}{(\tilde{n}_{15} + \bar{y}_I)^3}\varphi_2^2\right]\varphi_4$,

$$\rho_2 = \frac{2\tilde{n}_{14}\tilde{n}_{15}}{(\tilde{n}_{15} + \bar{y}_I)^3}\varphi_2^2 + [2\tilde{n}_5\tilde{n}_6(2\bar{y}_S) + \tilde{n}_{13}]\varphi_2 + 2\tilde{n}_5\tilde{n}_6\bar{y}_I,$$

$$\varphi_1 = -3\tilde{n}_5\tilde{n}_6\bar{y}_S^2 + 2\tilde{n}_5(\alpha_4 - \tilde{n}_6\bar{y}_I)\bar{y}_S + (\tilde{n}_5\tilde{n}_6\tilde{n}_7 - \tilde{n}_{13})\bar{y}_I - \frac{\tilde{n}_{11}\tilde{n}_{12}}{(\tilde{n}_{12} + \bar{y}_S)^2} - (\tilde{n}_9 + \tilde{n}_5\tilde{n}_6), \varphi_2 = -\frac{\tau_{32}}{\tau_{33}},$$

$$\varphi_4 = -\frac{\tau_{23}}{\tau_{33}}, \tau_{23} = \tilde{n}_5\tilde{n}_6\bar{y}_S^2 + (\varepsilon_2)\bar{y}_S + \frac{\tilde{n}_{14}\tilde{n}_{15}}{(\tilde{n}_{15} + \bar{y}_I)^2}, \tau_{32} = \tilde{n}_{13}\bar{y}_I,$$

$$\tau_{33} = \tilde{n}_{13}\bar{y}_S - \tilde{n}_{16} - \frac{\tilde{n}_{14}\tilde{n}_{15}}{(\tilde{n}_{15} + \bar{y}_I)^2}.$$

System 2 at the (EP) $E_7 = (0, \bar{y}_S, \bar{y}_I)$ with the parameter $\tilde{n}_{10} = \varphi_1 - \frac{\tau_{23}\tau_{32}}{\tau_{33}}$ has a (SNB) but (TB and PB) cannot occur.

Proof: From the Jacobian matrix J_7 given in **Equation 19** as in ³² the **System 2** at $\tilde{n}_{16} = \tilde{n}_{16} = \varphi_1 - \frac{\tau_{23}\tau_{32}}{\tau_{33}}$, the eigenvalue of (EP) E_7 is zero, as stated by $(\lambda_{7y_S} = 0)$ then J_3 with $\tilde{n}_{16} = \tilde{n}_{16}$ becomes

$$\bar{J}_7 = [\tau_{ij}]_{3 \times 3}, \text{ where } \tau_{11} = -\left(\frac{\bar{y}_S}{\tilde{n}_2} + \delta_1\right), \tau_{21} = \frac{(\tilde{n}_8\bar{y}_S)}{\tilde{n}_2}, \bar{\tau}_{22} = \frac{\tau_{23}\tau_{32}}{\tau_{33}}, \tau_{31} = \tau_{12} = \tau_{13} = 0.$$

Now, let $\check{v}^{[7]} = (\check{v}_1^{[7]}, \check{v}_2^{[7]}, \check{v}_3^{[7]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{7y_S} = 0$ for $\bar{J}_7$. Thus $(\bar{J}_7 - \lambda_{7y_S}I)\check{v}^{[7]} = 0$, that gives $\check{v}^{[7]} = (0, \check{v}_2^{[7]}, \varphi_2\check{v}_2^{[7]})^T$ where $\check{v}_3^{[3]} \neq 0$ is any real number. Let $\mathbb{W}^{[7]} = (\mathbb{W}_1^{[7]}, \mathbb{W}_2^{[7]}, \mathbb{W}_3^{[7]})^T$ For $\lambda_{7y_S} = 0$, be the eigenvector of $\bar{J}_7^T$. Consequently, we have $(\bar{J}_7^T - \lambda_{7y_S}I)\mathbb{W}^{[7]} = 0$ is obtained by solving this equation we get $\mathbb{W}^{[7]} = (\varphi_3\mathbb{W}_2^{[7]}, \mathbb{W}_2^{[7]}, \varphi_4\mathbb{W}_2^{[7]})^T$, where $\varphi_3 = -\frac{\tau_{21}}{\tau_{11}}, \mathbb{W}_2^{[7]} \neq 0$ is any real number.

Now, consider $\frac{\partial f}{\partial \tilde{n}_{10}} = f_{\tilde{n}_{10}}(X, \tilde{n}_{10}) = \left(\frac{\partial f_1}{\partial \tilde{n}_{10}}, \frac{\partial f_2}{\partial \tilde{n}_{10}}, \frac{\partial f_3}{\partial \tilde{n}_{10}}\right)^T = (0, -y_S, 0)^T$.

So, $f_{\tilde{n}_{10}}(E_7, \tilde{n}_{10}) = (0, -\bar{y}_S, 0)^T$ and hence $(\mathbb{W}^{[7]})^T f_{\tilde{n}_{10}}(E_7, \tilde{n}_{10}) = -\bar{y}_S \mathbb{W}_2^{[7]} \neq 0$. Now, by substituting $\check{v}^{[7]}$ in **Equation 4** we get

$$(\mathbb{W}^{[7]})^T [D^2 f_{\tilde{n}_{10}}(E_7, \tilde{n}_{10})(\check{v}^{[7]}, \check{v}^{[7]})] = [\rho_1 \neq \rho_2] \check{v}_2^{[7]^2} \mathbb{W}_2^{[7]} \neq 0$$

where $\mathbb{A}_1, \mathbb{A}_2$ in the state of theorem. under condition (21) which is given in ³² and conditions (17,18 and 19). Therefore, by using "Sotomayor's theorem" **System (2)** has a (SNB) at E_7 with the parameter $\tilde{n}_{16} = \tilde{n}_{16}$ but a (TB and PB) cannot be occurs at E_7 .

2.1.5. Theorem: Using the conditions (26, 27, 28, and 30) listed in ³², let's assume that the following condition is true:

$$\tilde{n}_{13}y_S^* > \frac{\tilde{n}_{14}\tilde{n}_{15}}{(\tilde{n}_{15} + y_I^*)^2}, \quad (20)$$

$$\delta_1 > \frac{\rho_{11}\rho_{23}\rho_{32}}{\rho_{11}\rho_{22} - \rho_{12}\rho_{21}}, \quad (21)$$

$$y_S^* < 1, \quad (22)$$

$$\tilde{n}_5 \tilde{n}_6 \tilde{n}_7 y_S^* (1 - y_S^*) + \frac{\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I^*)^2} > \tilde{n}_{13} y_S^*, \quad (23)$$

$$\mathbf{A}_1 \neq \mathbf{A}_2. \quad (24)$$

$$\text{where } \mathbf{A}_1 = -2 \left[\left(1 + \frac{\tilde{n}_2 y_S}{(\tilde{n}_2 + x)^3} \right) \mathfrak{A}_2^2 - \frac{\tilde{n}_2}{(\tilde{n}_2 + x)^2} \mathfrak{A}_2 \mathfrak{A}_3 \right] \mathfrak{A}_4 + [2\tilde{n}_5 (1 + \tilde{n}_6 \tilde{n}_7) \mathfrak{A}_3^2 + 2[\tilde{n}_5 \tilde{n}_6 \tilde{n}_7] \mathfrak{A}_3] \mathfrak{A}_3 + 2[\tilde{n}_{13} \mathfrak{A}_3 + \frac{\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I)^3}] \mathfrak{A}_3,$$

$$\mathbf{A}_2 = 2[(-3x - (\tilde{n}_1 + \tilde{n}_3)) \mathfrak{A}_2^2] \mathfrak{A}_4 - \left[-\frac{2\tilde{n}_2 \tilde{n}_8 y_S}{(\tilde{n}_2 + x)^3} \mathfrak{A}_2^2 + \frac{\tilde{n}_8}{(\tilde{n}_2 + x)^2} (\tilde{n}_2 + 1) \mathfrak{A}_2 \mathfrak{A}_3 - 2\tilde{n}_5 \tilde{n}_6 (3 + y_I) \mathfrak{A}_3^2 - [4\tilde{n}_5 \tilde{n}_6 y_S + \tilde{n}_{13}] \mathfrak{A}_3 - \frac{2\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I)^3} \right] \mathfrak{A}_3,$$

$$\mathfrak{A}_2 = \frac{\rho_{12} \rho_{23}}{\rho_{11} \rho_{22} - \rho_{12} \rho_{21}}, \mathfrak{A}_3 = -\frac{\rho_{11} \rho_{23}}{\rho_{11} \rho_{22} - \rho_{12} \rho_{21}}, \mathfrak{A}_4 = \frac{\rho_{21} \rho_{23}}{\rho_{11} \rho_{22} - \rho_{12} \rho_{21}}, \rho_{11} = -3x^{*2} - 2\delta_2 x^* - \frac{\tilde{n}_2 y_S^*}{(\tilde{n}_2 + x^*)^2} - (\delta_1), \rho_{12} = \frac{-x^*}{\tilde{n}_2 + x^*}, \rho_{21} = \frac{\tilde{n}_2 \tilde{n}_8 y_S^*}{(\tilde{n}_2 + x^*)^2}, \rho_{22} = -3\tilde{n}_5 \tilde{n}_6 y_S^{*2} + 2\tilde{n}_5 (\alpha_4 - \tilde{n}_6 y_I^*) y_S^* + (\varepsilon_2) y_I^* + \frac{\tilde{n}_8 x^*}{\tilde{n}_2 + x^*} - \frac{\tilde{n}_{11} \tilde{n}_{12}}{(\tilde{n}_{12} + y_S^*)^2} - (\delta_4), \rho_{23} = -\tilde{n}_5 \tilde{n}_6 \tilde{n}_7 y_S^{*2} + \varepsilon_2 y_S^* + \frac{\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I^*)^2}, \rho_{32} = \tilde{n}_{13} y_S^*, \mathfrak{A}_1 = \tilde{n}_{13} y_S^* - \frac{\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I^*)^2}.$$

System 2 at the (EP) $E_8 = (x^*, y_S^*, y_I^*)$ with the parameter $\tilde{n}_{16} = \mathfrak{A}_1 - \frac{\rho_{11} \rho_{23} \rho_{32}}{\rho_{11} \rho_{22} - \rho_{12} \rho_{21}}$ has a (TP and PT) but (SNB) cannot occur.

Proof: The **System 2** at the (EP) E_8 has eigenvalue ($\lambda_{8y_I} = 0$) equal to zero at $\tilde{n}_{16}^* = \tilde{n}_{16} = \mathfrak{A}_1 - \frac{\rho_{11} \rho_{23} \rho_{32}}{\rho_{11} \rho_{22} - \rho_{12} \rho_{21}}$ from the (JM) J_8 provided in **Equation 23** as in ³² then J_8 with $\tilde{n}_{16}^* = \tilde{n}_{16}$ is now $J_8^* = [\rho_{ij}]_{3 \times 3}$, where $\rho_{33} = \rho_{33}^* = \frac{\rho_{11} \rho_{23} \rho_{32}}{\rho_{11} \rho_{22} - \rho_{12} \rho_{21}}$.

Now, let $\check{v}^{[8]} = (\check{v}_1^{[8]}, \check{v}_2^{[8]}, \check{v}_3^{[8]})^T$ be the eigenvalue's associated eigenvector $\lambda_{8y_I} = 0$ for J_8^* .

Thus $(J_8^* - \lambda_{8y_I} I) \check{v}^{[8]} = 0$, that gives $\check{v}^{[8]} = (\mathfrak{A}_2 \check{v}_3^{[8]}, \mathfrak{A}_3 \check{v}_3^{[8]}, \check{v}_3^{[8]})^T$ where, $\check{v}_3^{[8]} \neq 0$ is any real number. Let $\mathbb{W}^{[8]} = (\mathbb{W}_1^{[8]}, \mathbb{W}_2^{[8]}, \mathbb{W}_3^{[8]})^T$ be its eigenvector J_8^{*T} for $\lambda_{8y_I} = 0$. Next, we get $(J_8^{*T} - \lambda_{8y_I} I) \mathbb{W}^{[8]} = 0$ then by solving this equation for $\mathbb{W}^{[8]}$ we get $\mathbb{W}^{[8]} = (\mathfrak{A}_4 \mathbb{W}_3^{[8]}, \mathfrak{A}_3 \mathbb{W}_3^{[8]}, \mathbb{W}_3^{[8]})^T$, where $\mathbb{W}_3^{[8]} \neq 0$ is any real number.

Now, consider $\frac{\partial f}{\partial \tilde{n}_{16}} = f_{\tilde{n}_{16}}(X, \tilde{n}_{16}^*) = \left(\frac{\partial f_1}{\partial \tilde{n}_{16}}, \frac{\partial f_2}{\partial \tilde{n}_{16}}, \frac{\partial f_3}{\partial \tilde{n}_{16}} \right)^T = (0, 0, -y_S)^T$.

So, $f_{\tilde{n}_{16}}(E_8, \tilde{n}_{16}^*) = (0, 0, -y_I^*)^T$ and hence $(\mathbb{W}^{[8]})^T f_{\tilde{n}_{16}}(E_8, \tilde{n}_{16}^*) = -y_I^* \mathbb{W}_3^{[8]}$.

Now, by substituting $\check{v}^{[8]}$ in **Equation 4** we get

$$(\mathbb{W}^{[8]})^T [D^2 f_{\tilde{n}_{16}}(E_8, \tilde{n}_{16}^*) (\check{v}^{[8]}, \check{v}^{[8]})] = [\mathbf{A}_1 \neq \mathbf{A}_2] \check{v}_3^{[8]^2} \mathbb{W}_3^{[8]} \neq 0$$

where $\mathbf{A}_1, \mathbf{A}_2$ in the state of theorem. under condition (26,27,28 and 30) which is given in ³² and conditions (20,21,22,23 and 24). Therefore, in **System 2**, a (SNB) with the quantity $\tilde{n}_{16}^* = \tilde{n}_{16}$ may be obtained using "Sotomayor's theorem," however a (TP and PB) can't occur at E_8 .

2.2. Hopf Bifurcation Analysis:

In this section, the following theorem demonstrates that an application to the Hopf bifurcation (HB) ³⁴ for local bifurcation (LB) is suitable when there is a chance of the (HB) occurring close to the positive (EP) E_8 of the **System 2**.

2.2.1.Theorem: Suppose the following circumstances exist and that conditions (25,26,27,28, and 30) listed in ³² are true:

$$Z_2 < 0, \quad (25)$$

$$\tilde{n}_{13} y_S^* > \frac{\tilde{n}_{14} \tilde{n}_{15}}{(\tilde{n}_{15} + y_I^*)^2}, \quad (26)$$

$$\rho_{11} \rho_{22} - \rho_{12} \rho_{21} < -\frac{\rho_{22} \rho_{23} \rho_{32}}{\rho_{11} + \rho_{22}}, \quad (27)$$

$$\rho_{11} + \rho_{22} > \frac{\rho_{23}\rho_{32}}{\rho_{11} + \rho_{22}}, \quad (28)$$

$$2[(\rho_{22} + \rho_{33})(\rho_{11} - I_1(\tilde{n}_{16}^*)) - \rho_{12}\rho_{21}]I_2(\tilde{n}_{16}^*) \neq 0. \quad (29)$$

Where $\mathcal{Z}_2 = F_1^2 - Q_1 F_1 + Q_2$,

$$\text{With } Q_1 = \rho_{11} + \rho_{22} - \frac{\rho_{23}\rho_{32}}{\rho_{11} + \rho_{22}}, Q_2 = \rho_{11}\rho_{22} - \rho_{12}\rho_{21} + \frac{\rho_{22}\rho_{23}\rho_{32}}{\rho_{11} + \rho_{22}}, F_1 = \tilde{n}_{13}y_S^* - \frac{\tilde{n}_{14}\tilde{n}_{15}}{(\tilde{n}_{15} + y_I^*)^2}.$$

Then at the parameter value $\tilde{n}_{16}^* = \tilde{n}_{16}$ **System 2** has a (HB) near the positive point E_8 .

Proof: Consider the characteristic equation of **System 2** at E_8 which is given in ³². Next, using the (HB) theorem for $n=3$, we need to select a parameter, let's say $(\tilde{n}_{16}^*)$, to ensure that the necessary and sufficient conditions for (HB) to occur fulfill that $I_i(\tilde{n}_{16}^*) > 0$, $i = 1; 2$ and $\Delta_1(\tilde{n}_{16}^*) = I_1I_2 - I_3 > 0$ which is given in ³².

Straight forward computation gives that $I_i(\tilde{n}_{16}^*) > 0$; $i = 1, 2$ and $\Delta_1(\tilde{n}_{16}^*) = I_1I_2 - I_3 > 0$ provided that conditions (25,26,27,28 and 30) which is given in ³² are hold. On the other hand, it is observed that $\Delta_1 = 0$ gives

$$\tilde{n}_{16}^{*2} + \mathcal{Z}_1 \tilde{n}_{16}^* + \mathcal{Z}_2 = 0 \quad (30)$$

where $\mathcal{Z}_1 = 2F_1 - Q_1$,

Where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ in the state of theorem, then by using Descartes rule of sign, **Equation 30** has a unique positive root $(\tilde{n}_{16}^*)$ if in addition to requirements (25,26,27,28 and 30) which is given in ³² and one of the sets of condition (25,26,27 and 28).

Now, at $\tilde{n}_{16} = \tilde{n}_{16}^*$ the characteristic equation $\lambda^3 + I_1\lambda^2 + I_2\lambda + I_3 = 0$ which is given in ³² can be written as $P(\lambda) = (\lambda + I_1)(\lambda^2 + I_2) = 0$,

which have two roots $\lambda_1 = -I_1$ and $\lambda_{2,3} = \pm i\sqrt{I_2}$

Clearly, at $\tilde{n}_{16} = \tilde{n}_{16}^*$ there are one eigenvalue (λ_1) which is real and negative and (two) pure imaginary eigenvalues $(\lambda_{2,3})$. Now for all values of $\tilde{n}_{16}$ in the vicinity of $\tilde{n}_{16}^*$ the roots in general of the subsequent form $\lambda_{2,3} = \omega_1(\tilde{n}_{16}) \pm i\omega_2(\tilde{n}_{16})$.

Now, according to verify the transversality condition, we must prove that

$$\Theta^*(\tilde{n}_{16}^*) \Psi^*(\tilde{n}_{16}^*) + \Gamma^*(\tilde{n}_{16}^*) \Phi^*(\tilde{n}_{16}^*) \neq 0,$$

Note that for $\tilde{n}_{16} = \tilde{n}_{16}^*$ we have $\omega_1 = 0$ and $\omega_2 = \sqrt{I_2}$, substituting the value of ω_2 gives the following simplifications:

$$\Psi^*(\tilde{n}_{16}^*) = -2I_2(\tilde{n}_{16}^*), \Phi^*(\tilde{n}_{16}^*) = 2I_1(\tilde{n}_{16}^*)\sqrt{I_2}(\tilde{n}_{16}^*),$$

$$\Theta^*(\tilde{n}_{16}^*) = -y_S^*[\rho_{23}\rho_{32} + \rho_{22}\rho_{33} + I_2(\tilde{n}_{16}^*)], \Gamma^*(\tilde{n}_{16}^*) = -y_S^*(\rho_{22} + \rho_{33})\sqrt{I_2}(\tilde{n}_{16}^*),$$

so $\Theta^*(\tilde{n}_{16}^*) \Psi^*(\tilde{n}_{16}^*) + \Gamma^*(\tilde{n}_{16}^*) \Phi^*(\tilde{n}_{16}^*) = 2[(\rho_{22} + \rho_{33})(\rho_{11} - I_1(\tilde{n}_{16}^*)) - \rho_{12}\rho_{21}]I_2(\tilde{n}_{16}^*) \neq 0$. condition (29) such that we may determine that the (HB) occurs at the parameter $\tilde{n}_{16}^* = \tilde{n}_{16}$ around the (EP) E_8 .

3. Numerical Analyses:

The MATLAB program is employed to numerically analyze the dynamic behavior of **System 2** to confirm our analytical results and investigate the effect of modifying the value of every parameter with regard to system's dynamic behavior. It should be observed that **System 2** has a Global stability analysis (GAS) positive equilibrium point (PEP), as shown in **Figure 1**, with the subsequent Hypothetical parameter configuration meeting the stability conditions of the (PEP).

$$\left. \begin{aligned} \tilde{n}_1 &= 2.8, \tilde{n}_2 = 3.5, \tilde{n}_3 = 0.002, \tilde{n}_4 = 0.001, \tilde{n}_5 = 3, \tilde{n}_6 = 0.22, \\ \tilde{n}_7 &= 0.01, \tilde{n}_8 = 0.5, \tilde{n}_9 = 0.3, \tilde{n}_{10} = 0.008, \tilde{n}_{11} = 0.01, \\ \tilde{n}_{12} &= 0.1, \tilde{n}_{13} = 0.6, \tilde{n}_{14} = 0.2, \tilde{n}_{15} = 0.2, \tilde{n}_{16} = 0.001. \end{aligned} \right\} \quad (31)$$

Obviously, **Figure 1** shows that the **System 2** has a (GAS) as the solution of the **System 2** approach asymptotically to the (PEP) $E_8 = (2.785, 0.127, 2.468)$. Now, to discuss the effect of the parameter values on the dynamical behaviour of the system, we change one of the parameters at a time while keeping the other parameters as shown and the results obtained are shown in **Table 2**.

Furthermore, The effect of $\tilde{n}_{16}$ was varied in the range $0 < \tilde{n}_{16} < 0.075$, it was found that **System 2** approaches to E_8 ; additionally, in the range $0.075 \leq \tilde{n}_{16} < 0.1$ was studied, that approaches to a periodic attractor in xy_Sy_I -plane and in the range $0.15 < \tilde{n}_{16} \leq 0.72$ was studied, that approaches to a periodic attractor in y_Sy_I -plane, however increasing $\tilde{n}_{16}$ as in the range $0.72 < \tilde{n}_{16} \leq 0.74$, the system approaches to E_7 , as illustrated in **Figure 2**.

Table 2. The Dynamical Behavior of System 2

Range of Parameter	Stable Point	Bifurcation	Persistence
$0 < \tilde{n}_1 < 0.594$, $0.594 \leq \tilde{n}_1 \leq 1.318$, $2.732 \leq \tilde{n}_1 \leq 5.6$	E_8 E_7 E_8	0.594	Persists Not Persists Persists
$3.79 \leq \tilde{n}_2$	E_8		Persists
$0 < \tilde{n}_3 \leq 0.012$	E_8		Persists
$0 < \tilde{n}_4 \leq 0.013$	E_8		Persists
$0 < \tilde{n}_5 < 0.183$ $0.183 \leq \tilde{n}_5 \leq 0.3$ $0.3 < \tilde{n}_5 < 3.295$	E_1 Periodic attractor in xy_Sy_I -plane E_8	0.183 0.3	Not Persists Persists Persists
$0.21 \leq \tilde{n}_6 \leq 0.984$ $0.984 < \tilde{n}_6 \leq 4.786$	E_8 E_5	0.984	Persists Not Persists
$0 < \tilde{n}_7 \leq 0.196$ $0.196 < \tilde{n}_7 \leq 0.292$ $0.292 < \tilde{n}_7 \leq 0.999$	E_8 Periodic attractor in xy_Sy_I -plane E_1	0.196	Persists Persists Not Persists
$0 < \tilde{n}_8 < 1$	E_8		Persists
$0.156 \leq \tilde{n}_9 \leq 0.7$ $0.7 < \tilde{n}_9 \leq 1$	E_8 Periodic attractor in xy_Sy_I -plane	0.7	Persists Persists
$0 < \tilde{n}_{10} < 0.42$ $0.4 \leq \tilde{n}_{10} < 0.738$ $0.738 \leq \tilde{n}_{10} < 2.796$	E_8 Periodic attractor in xy_Sy_I -plane E_1	0.4 0.738	Persists Persists Not Persists
$0 \leq \tilde{n}_{11} \leq 0.1$ $0.1 < \tilde{n}_{11} \leq 0.213$ $0.213 \leq \tilde{n}_{11} \leq 0.299$	E_8 Periodic attractor in xy_Sy_I -plane E_1	0.1 0.213	Persists Persists Not Persists
$0 < \tilde{n}_{12}$	E_8		Persists
$0 < \tilde{n}_{13} < 0.046$ $0.556 \leq \tilde{n}_{13} < 11.268$	E_3 E_8		Not Persists Persists
$0 < \tilde{n}_{14} < 0.016$ $0.016 \leq \tilde{n}_{14} < 0.023$ $0.023 \leq \tilde{n}_{14} < 0.297$	E_1 Periodic attractor in xy_Sy_I -plane E_8	0.016 0.023	Not Persists Persists Persists
$0 < \tilde{n}_{15} \leq 4.86$ $4.86 < \tilde{n}_{15} < 4.958$ $4.958 \leq \tilde{n}_{15}$	E_8 Periodic attractor in xy_Sy_I -plane E_1	4.86 4.958	Persists Persists Not Persists
$0 < \tilde{n}_{16} < 0.075$ $0.075 \leq \tilde{n}_{16} \leq 0.1$ $0.15 \leq \tilde{n}_{16} < 0.72$ $0.72 \leq \tilde{n}_{16} < 0.74$	E_8 Periodic attractor in xy_Sy_I -plane Periodic attractor in y_Sy_I -plane E_7	0.075 0.72	Persists Persists Not Persists Not Persists

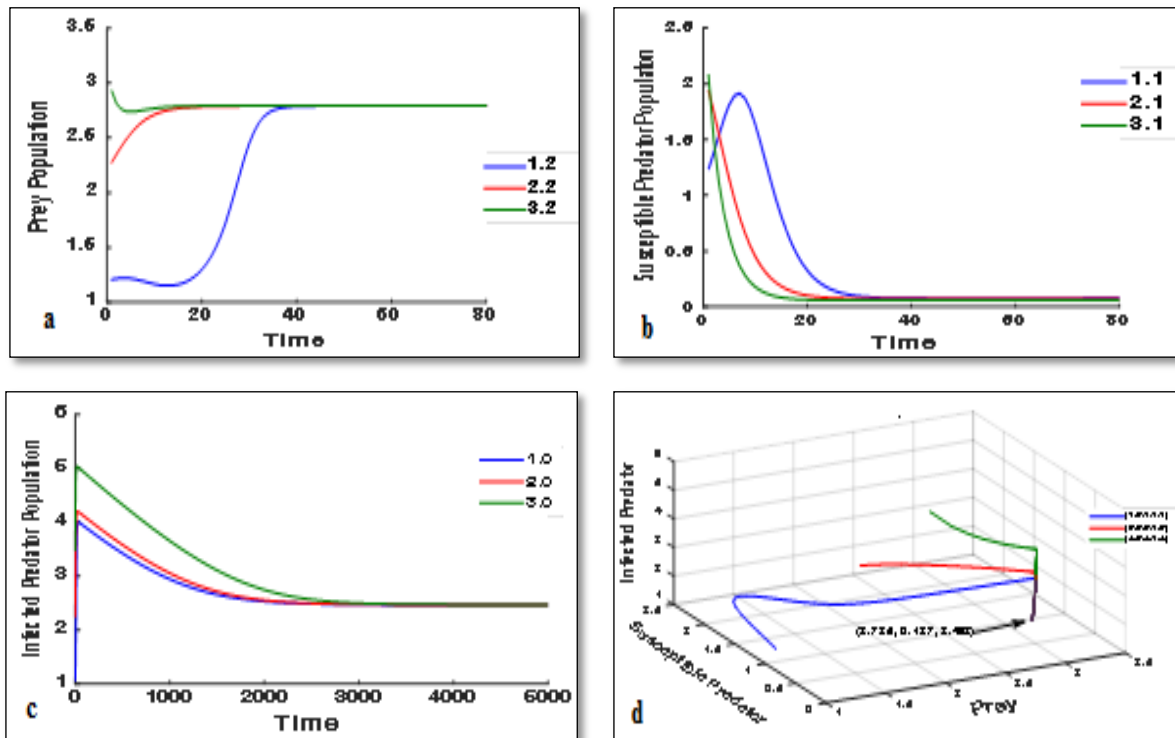


Figure 1. The paths taken by System 2, which began at three distinct starting locations $(1.2, 1.1, 1)$, $(2.2, 2.1, 2)$ and $(3.2, 3.1, 3)$ for the data given in 5×10^3 , a. trajectory of x with relation to time, b. trajectory of y_s as result of time, c. trajectory of y_i as a function of time, d. trajectory approaches to $E_8 = (2.785, 0.127, 2.468)$.

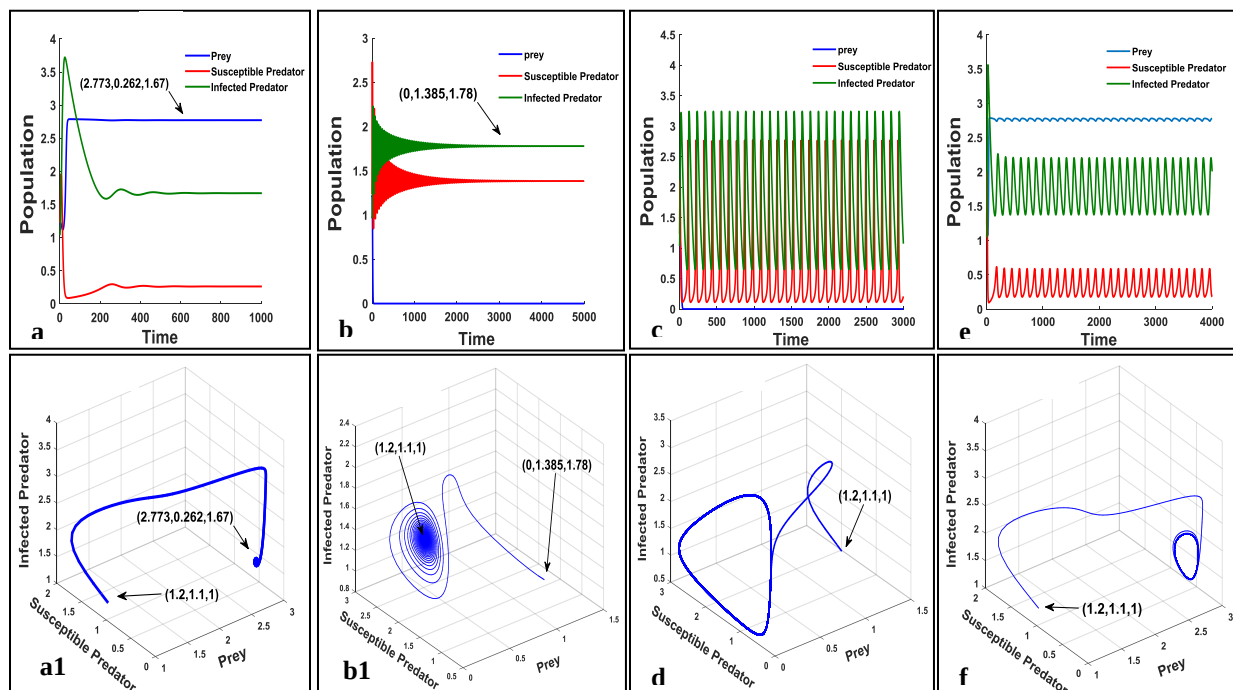


Figure 2. Time trend illustrating the solution of System 2 which approaches to $E_8 = (2.773, 0.262, 1.67)$ when $\tilde{n}_{16} = 0.05$, (a1) 3D phase portrait of (a), (b) Time series of the solution approaches to $E_7 = (0, 1.385, 1.78)$ when $\tilde{n}_{16} = 0.73$, (b1) 3D phase portrait of (b), (c) Time series of solution when $\tilde{n}_{16} = 0.2$ that approaches to a periodic attractor in $y_s y_i$ -plane, (d) 3D phase portrait of (c), (e) Time series of solution when $\tilde{n}_{16} = 0.09$ that approaches to a periodic attractor in $x y_s y_i$ -plane, (f) 3D phase portrait of (e).

4. Discussions

In this paper, the conditions for the occurrence of (LB) of an eco-toxicant-epidemiological model of the infected and susceptible predator-prey type with logistic growth and strong Allee effect, nonlinear harvesting of the Michaelis-Menten type in the susceptible predator, as well as treatment in the infected predator, were determined. It was noted that in the vicinity of the equilibrium points:

- At E_1, E_3 and E_5 **System 2** possesses a (TB and PB) at the external effort devoted to harvesting of susceptible predator ($\tilde{n}_{11}$) and the death rate of Infected predator ($\tilde{n}_{16}$).
- At E_7 and E_8 **System 2** possesses a (SNB) at the toxicity susceptible predator ($\tilde{\tilde{n}}_{10}$) and the death rate of Infected predator ($\tilde{n}_{16}^*$) respectively.

Additionally, inquiries regarding (HB) in proximity to E_8 is being conducted.

Ultimately, MATLAB was used to perform numerical simulations using three different starting points and a fictitious dataset provided by **Data 31**.

5. Conclusion

The **Data 31** presented in this research regarding the system, methodology utilized, and numerical results demonstrate that the parameters $\tilde{n}_i$, $i = 1, 5, 6, 7, 9, 10, 11, 14, 15, 16$ significantly affect the stability of **System 2**, while the parameters $\tilde{n}_i$, $i = 2, 3, 4, 8, 12, 13$ do not impact the stability of **System 2**, as the solutions consistently converge to the (PEP).

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Conflict of Interest

None conflict.

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